(Cont' from last lecture)

$F_2(z)$ is clearly integrable on $[-\pi, \pi]$

For $F_1(z)$, note that $1-\Phi_\delta(z)=0$ on $[-\frac{\delta}{2},\frac{\delta}{2}]$ &

$$|\sin\frac{z}{2}| \geq \sin\frac{\delta}{4} > 0 \quad \text{for } \frac{\delta}{2} \leq |z| \leq \pi.$$

$\Rightarrow$ $F_1(z)$ is also integrable on $[-\pi,\pi]$.

Therefore Riemann-Lebesgue lemma implies

$$\left.\begin{matrix} b_n(F_1) \\ a_n(F_2) \end{matrix}\right\} \longrightarrow 0 \text{ as } n \to \infty$$

$\therefore$ $\exists\, n_0 > 0$ s.t. $\left.\begin{matrix} |b_n(F_1)| < \frac{\varepsilon}{4} \\ |a_n(F_2)| < \frac{\varepsilon}{4} \end{matrix}\right.$ for $n \geq n_0$

$$\therefore \quad |II| \leq |b_n(F_1)| + |a_n(F_2)| < \frac{\varepsilon}{2}.$$

<u>Final Step</u>: By Steps 3, 4, & 5, we have

$\forall \varepsilon > 0$, $\exists\, n_0 > 0$ s.t.

$$\begin{aligned} |S_nf(x_0) - f(x_0)| &= |I + II| \\ &\leq |I| + |II| \leq \frac{4\delta L}{\pi} + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

$\therefore$ $S_nf(x_0) \to f(x_0)$ as $n \to \infty$ ✱

§1.4 Weierstrass Approximation Theorem

(Application of Thm 1.7)

Recall: A cts function defined on $[a,b]$ is piecewise linear if $\exists$ a partition $a=a_0<a_1<\cdots<a_n=b$ s.t. f is linear on each subinterval $[a_j, a_{j+1}]$.

Prop 1.11 Let f be a cts function on $[a,b]$. Then $\forall \varepsilon>0$, $\exists$ a cts, piecewise linear g with $g(a)=f(a)$, $g(b)=f(b)$ such that

$$\|f-g\|_\infty < \varepsilon$$

$$\left(\|f-g\|_\infty = \sup_{[a,b]} |f(x)-g(x)| \right)$$

Pf: f cts on closed interval $[a,b]$

$\Rightarrow$ f uniformly cts $[a,b]$

$\Rightarrow$ $\forall \varepsilon>0$, $\exists \delta>0$ s.t.

$$|f(x)-f(y)|<\frac{\varepsilon}{2}, \quad \forall\ |x-y|<\delta \quad (x,y\in[a,b])$$

Partition $[a,b]$ into subintervals $I_j=[a_j, a_{j+1}]$ s.t.

$$|I_j| = a_{j+1}-a_j < \delta, \quad \forall j.$$

Define $g(x) = \dfrac{f(a_{j+1})-f(a_j)}{a_{j+1}-a_j}(x-a_j)+f(a_j)$, $\forall x\in I_j$

Clearly $g(a_j) = f(a_j)$, $\forall j$. In particular $g(a) = f(a)$ & $g(b) = f(b)$, and $g(x)$ is piecewise linear on $[a,b]$.

Then $\forall x \subset I_j \subset [a,b]$

$$|f(x) - g(x)| = \left| f(x) - \frac{f(a_{j+1}) - f(a_j)}{a_{j+1} - a_j}(x - a_j) - f(a_j) \right|$$

$$\le |f(x) - f(a_j)| + \frac{|f(a_{j+1}) - f(a_j)|}{a_{j+1} - a_j}(x - a_j)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\therefore \quad \sup_{\cup I_j} |f(x) - g(x)| < \varepsilon \quad$ ie $\|f - g\|_\infty < \varepsilon$. #

<u>Terminology</u>: A <u>trigonometric polynomial</u> is of the form $P(\cos x, \sin x)$, where $P(x,y)$ is a polynomail of 2 variables.

<u>Note that</u> a trigonometric polynomial is a finite Fourier series and vice-versa. (Ex!)

[finite Fourier series: $a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$, $N < \infty$]

<u>Prop 1.12</u> Let f be cts function on $[0, \pi]$. Then $\forall \varepsilon > 0$, $\exists$ a trigonometric polynomial h s.t. $\|f - h\|_\infty < \varepsilon$

Pf: Extend f to $[-\pi, \pi]$ by

$$f(x) = \begin{cases} f(x), & x \in [0, \pi] \\ f(-x), & x \in [-\pi, 0] \end{cases}$$

Then this extension is cts on $[-\pi, \pi]$ & $f(\pi) = f(-\pi)$, hence extends to a 2π-periodically cts function on $\mathbb{R}$.

By Prop 1.11, $\forall \varepsilon > 0$, $\exists$ piecewise linear (cts) g on $[-\pi, \pi]$ s.t. $\|f - g\|_\infty < \frac{\varepsilon}{2}$ &

$$g(\pi) = f(\pi) = f(-\pi) = g(-\pi)$$

$\Rightarrow$ g extends to a piecewise linear 2π-periodic function $\tilde{g}$ on $\mathbb{R}$.

Clearly $\tilde{g}$ satisfies a Lip condition.

Then Thm 1.7 $\Rightarrow$ $\exists N > 0$ s.t.

$$\|g - S_N g\|_\infty < \frac{\varepsilon}{2} \quad \left(\begin{matrix} S_N g \to g \\ \text{uniformly} \end{matrix} \right)$$

Therefore $\|f - S_N g\|_\infty \leq \|f - g\|_\infty + \|g - S_N g\|_\infty$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2},$$

$\therefore$ $h = S_N g$ is the required trigonometric polynomial $\#$

Thm 1.13 (Weierstrass Approximate Theorem)

Let $f \in C[a,b]$. $\forall \varepsilon > 0$, $\exists$ a polynomial g s.t.

$$\|f - g\|_\infty < \varepsilon.$$

Pf: Consider $[a,b] = [0, \pi]$ first.

Extend f to $[-\pi, \pi]$ as in prop 1.12

$\forall \varepsilon > 0$, choose trigonometric polynomial

$h = p(\cos x, \sin x)$ s.t.

$$\|f - h\|_\infty < \frac{\varepsilon}{2}$$

Using the fact that $\begin{cases} \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\ \sin x = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} \end{cases}$

converge uniformly

$\exists N > 0$ s.t.

$$\left\| h(x) - p\left(\sum_{n=1}^{N} \frac{(-1)^n x^{2n}}{(2n)!}, \sum_{n=1}^{N} \frac{(-1)^n x^{2n-1}}{(2n-1)!} \right) \right\|_\infty < \frac{\varepsilon}{2}$$

Clearly $g(x) = p\left(\sum_{n=1}^{N} \frac{(-1)^n x^{2n}}{(2n)!}, \sum_{n=1}^{N} \frac{(-1)^n x^{2n-1}}{(2n-1)!} \right)$

is the required polynomial s.t. $\|f - g\|_\infty < \varepsilon$.

For general $[a,b]$, $\varphi(x) = f\left(\frac{b-a}{\pi}x + a\right) \in C[0,\pi]$

$\Rightarrow$ $\exists$ $q(x)$ polynomial s.t.

$$\|\varphi(x) - q(x)\|_\infty < \varepsilon \quad \text{on } [-\pi,\pi]$$

$\Rightarrow$ $q\left(\frac{\pi}{b-a}(x-a)\right)$ is the polynomial s.t.

$$\left\| f(x) - q\left(\frac{\pi}{b-a}(x-a)\right)\right\|_\infty < \varepsilon$$

§1.5 Mean Convergence of Fourier Series

Notation : $R[-\pi,\pi] = \{$ set of Riemann integrable (real) functions on $[-\pi,\pi]$ $\}$

Def (1) $\forall$ $f, g \in R[-\pi,\pi]$, the L^2-product (L^2 inner product) is given by

$$\boxed{\langle f, g\rangle_2 = \int_{-\pi}^{\pi} f(x)g(x)\,dx}$$

(Note: for cpx functions $\langle f, g\rangle_2 = \int_{-\pi}^{\pi} f\bar{g}$)

(2) The L^2-norm of $f \in R[-\pi,\pi]$ is $\|f\|_2 = \sqrt{\langle f, f\rangle_2}$

(3) The L^2-distance between $f, g \in R[-\pi,\pi]$ is

$\|f-g\|_2$

(4) We said that $f_n \to f$ in L^2-sense if

$$\|f_n - f\|_2 \to 0 \quad \text{as } n \to \infty.$$

$$\left(\text{i.e. } \lim_{n\to\infty} \int_{-\pi}^{\pi} (f_n - f)^2 \, dx = 0, \text{ "mean convergence"} \right)$$

Caution: L^2-norm & L^2-distance on $R[-\pi,\pi]$ are not really "norm" & "distance" in strict sense

as
$$\begin{cases} \|f\|_2 = 0 \not\Rightarrow f = 0 \text{ in } R[-\pi,\pi] \\ \|f-g\|_2 = 0 \not\Rightarrow f = g \text{ in } R[-\pi,\pi]. \end{cases}$$

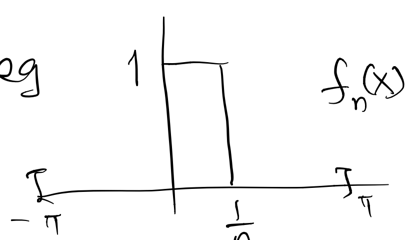
(We only have $\begin{cases} f = 0 \text{ almost everywhere.} \\ f = g \text{ almost everywhere.} \end{cases}$)

Note: It is not hard to show that $f_n \to f$ uniformly

$$\Rightarrow \|f_n - f\|_2 \to 0.$$

However $\|f_n - f\|_2 \to 0 \not\Rightarrow f_n \to f$ uniformly

eg



$$f_n(x) = \begin{cases} 1 & x \in [0, \frac{1}{n}] \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|f_n\|_2^2 = \int_{-\pi}^{\pi} f_n^2 = \frac{1}{n} \to 0 \quad \therefore f_n \to 0$ in L^2-sense

But $f_n \not\to 0$ uniformly. In fact $f_n(x) \to \begin{cases} 1 & \text{if } x=0 \\ 0 & \text{otherwise.} \end{cases}$

Application to Fourier series:

Consider the functions on $[-\pi, \pi]$

$$\begin{cases} \varphi_0 = \frac{1}{\sqrt{2\pi}} & (\text{const. function}) \\ \varphi_n = \frac{1}{\sqrt{\pi}} \cos nx & \\ \psi_n = \frac{1}{\sqrt{\pi}} \sin nx & \end{cases} \quad (n \geq 1)$$

Then

$$\begin{cases} \langle \varphi_m, \varphi_n \rangle_2 = \begin{cases} 1 & \text{if } m=n \\ 0 & \text{if } m \neq n \end{cases} \\ \langle \varphi_m, \psi_n \rangle_2 = 0 \quad \forall m, n \\ \langle \psi_m, \psi_n \rangle_2 = \begin{cases} 1 & \text{if } m=n \\ 0 & \text{if } m \neq n \end{cases} \end{cases}$$

$\therefore \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \sin nx \right\}_{n=1}^{\infty}$ can be regarded as an "orthonormal basis" in $R[-\pi, \pi]$

Notation We denote

$$E_N \stackrel{\text{def}}{=} \operatorname{span}\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \sin nx \right\}_{n=1}^{N}$$

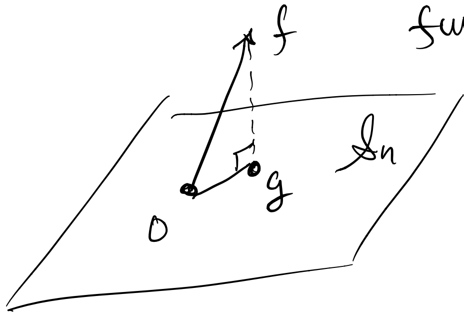
$= (2N+1)$ dim'l vector subspace of $R[-\pi, \pi]$ spanned by the 1^{st} $(2N+1)$ trigonometric functions.

$(\dim E_N = 2N+1)$

In general, if we have an orthonormal set (or orthonormal family) $\{\phi_n\}_{n=1}^{\infty}$ in $R[-\pi,\pi]$ $(\langle \phi_n, \phi_m \rangle_2 = \delta_{mn})$,

we set $\quad \mathcal{S}_n = \operatorname{span}\langle \phi_1, \cdots, \phi_n \rangle$

$= n$-dim'l subspace spanned the 1^{st} n functions in the orthonormal set.



Then $\forall f \in R[-\pi,\pi]$, we consider the minimization problem

$$\inf\{\|f-g\|_2 : g \in \mathcal{S}_n\}$$

Prop 1.14 The unique minimizer of $\inf\limits_{g \in \mathcal{S}_n} \|f-g\|_2$ is attained at the function

$$g = \sum_{k=1}^{n} \langle f, \phi_k \rangle_2 \phi_k \in \mathcal{S}_n$$

Pf: Note that minimize $\|f-g\|_2 \Leftrightarrow \|f-g\|_2^2$ minimize

Then $\forall g \in \mathcal{S}_n$, $g = \sum_{k=1}^{n} \beta_k \phi_k$ &

$$\|f-g\|_2^2 = \int_{-\pi}^{\pi} \left| f - \sum_{k=1}^{n} \beta_k \phi_k \right|^2 \overset{\text{regarded}}{=\!=} \underset{\Phi(\beta)}{\overset{\|}{\Phi(\beta_1, \cdots, \beta_n)}}$$

We first need to show that $\Phi(\beta_1, \cdots, \beta_n) \to \infty$ as $\|\beta\| \to +\infty$.
$\left(\sqrt{\beta_1^2 + \cdots + \beta_n^2}\right)$

$$\Phi(\beta) = \int_{-\pi}^{\pi} \left(f - \sum_{k=1}^{n} \beta_k \phi_k\right)^2$$

$$= \left(\int_{-\pi}^{\pi} f^2\right) - 2 \sum_{k=1}^{n} \left(\frac{\beta_k}{\sqrt{2}}\right)\left(\sqrt{2} \langle f, \phi_k \rangle_2\right) + \sum_{k=1}^{n} \beta_k^2$$

$$\left(2ab \le a^2 + b^2\right) \ge \left(\int_{-\pi}^{\pi} f^2\right) - \sum_{k=1}^{n} \left(\frac{\beta_k^2}{2} + 2\langle f, \phi_k \rangle_2^2\right) + \sum_{k=1}^{n} \beta_k^2$$

$$= \left(\int_{-\pi}^{\pi} f^2\right) - 2 \sum_{k=1}^{n} \langle f, \phi_k \rangle_2^2 + \frac{1}{2} \sum_{k=1}^{n} \beta_k^2$$

$$\longrightarrow +\infty \quad \text{as } \|\beta\| = \left(\sum_{k=1}^{n} \beta_k^2\right)^{1/2} \to +\infty.$$

$\therefore$ $\Phi(\beta)$ attains a minimum at some finite point $\beta = (\beta_1, \cdots, \beta_n)$.

Then easy calculus $\Rightarrow$ the unique minimum is given by
$\beta_k = \langle f, \phi_k \rangle$, $\forall k = 1, \cdots, n$

#

Notes : (1) The minimizer $g = \sum_{k=1}^{n} \langle f, \phi_k \rangle \phi_k$ of $\|f-g\|_2$ over S_n is called the orthogonal projection of f on S_n & denoted by $P_n f$.

(2) $\text{dist}(f, S_n) \left(= \inf \{ \text{dist}(f,g) : g \in S_n \} \right)$

$= \|f - P_n f\|_2$