MATH 2230A

Tutorial 9 9-11-2016

Topics: ∘ Liouville's theorem and Maximum modulus principle.

Thm: (Liouville's thm)

- Every bounded entire function f is constant.

Thm: (Maximum Modulus Principle)

- If f is analytic in a domain D and $\exists z_0 \in D$ s.t. $|f(z)| \leq f(z_0)$ $\forall z \in D$, then f is a constant.

Cor: If f is continuous on a closed and bounded region R and it is analytic and non-constant in the interior of R, then the max. value of $|f(z)|$ must be on the boundary of R.

Example: 1) If f is an entire function s.t. $|\operatorname{Im} f(z)| \leq M < +\infty$ $\forall z \in \mathbb{C}$, then f must be constant.

Ans: Consider the function $g(z) = e^{-if(z)}$.

Note that
$$|g(z)| = |e^{-if(z)}|$$
$$= e^{\operatorname{Re}(-if(z))}$$
$$= e^{\operatorname{Im} f(z)}$$
$$\leq e^{M} \quad \text{since } |f(z)| \leq M.$$

Since $g(z)$ is an entire function, by Liouville's thm, g must be a constant. Hence f must also be constant.

2) Show that there does not exist non-constant entire function $f(z)$ satisfying $(*)$:

$$(*): \begin{cases} f(z+1) = f(z) \\ f(z+i) = f(z) \end{cases} \quad \forall z \in \mathbb{C}.$$

Ans: It is equivalent to show that every entire function $f(z)$ satisfying $(*)$ is a constant function.

Let $D = \{x+iy \mid x, y \in [0,1]\}$, which is closed and bounded.
Since f is entire, it is continuous.
Since every continuous function on a closed and bounded set must be bounded, f is bounded on D.
Moreover, $\forall z \in \mathbb{C}, \exists k_1, k_2 \in \mathbb{Z}$ s.t. $z - k_1 - k_2 i \in D$.
As a result, since $f(z-k_1-k_2 i) = f(z)$ and f is bounded on D, f must be bounded on $\mathbb{C}$.
By Liouville's thm, it must be constant.

Remark: If $f: \mathbb{C} \to \mathbb{C}$ is analytic and satisfies $(*)$, we can think of it as an analytic function defined on the torus T^2. This question shows that every analytic function on T^2 must be constant.

3) Show that $\nexists$ entire function f s.t. $|f(z)| = e^{|z|}$ $\forall z \in \mathbb{C}$.

Ans: Suppose such function f exists.
Then $|f(z)| = e^{|z|} > 0$. Hence $f(z) \neq 0$. $\forall z \in \mathbb{C}$.
Consider the function $g(z) = \dfrac{1}{f(z)}$, which is entire.

Moreover, $|g(z)| = \dfrac{1}{|f(z)|} = e^{-|z|} < e^0 = 1$

By Liouville's thm, g must be constant.
If g is constant, then $f(z) = \dfrac{1}{g(z)}$ is also constant.

But then $|f(z)| = \text{constant} \neq e^{|z|}$ in general, contradiction.
Hence such f does not exist.

↙ I haven't discussed this in the tutorial

4) Find the max. pt. and max. value of the function $|f(z)|$ on $B_R(0)=\{z \mid |z|\le R\}$, where $f(z)=z^2+2iz$.

Ans: By max. principle, the max. pt. lies on the boundary $\partial B_R(0) = C_R(0)=\{z \mid |z|=R\}$.

When $z\in C_R(0)$, $z=Re^{i\theta}$, $\theta\in[0,2\pi]$,

$$
\begin{aligned}
|f(z)| &= |z^2+2iz| \\
&= |z|\,|z+2i| \\
&= R\,|Re^{i\theta}+2i| \\
&= R\,|R\cos\theta + (R\sin\theta+2)i|
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\ |f(z)|^2 &= R^2(R^2\cos^2\theta + R^2\sin^2\theta + 4R\sin\theta+4) \\
&= R^2(R^2+4R\sin\theta+4).
\end{aligned}
$$

$|f(z)|$ is max. $\Leftrightarrow |f(z)|^2$ is max $\Leftrightarrow \sin\theta=1 \Leftrightarrow \theta=\frac{\pi}{2}$

$\therefore$ Max. pt. $= Ri$

Max. value $=\sqrt{R^2(R^2+4R+4)} = R(R+2)$.

5) Show that the function $f(r)=\sup_{|z|=r}|g(z)|$ is an increasing function, where g is entire.

Ans: Let $0\le r_1\le r_2$.

Consider the set $B_{r_1}(0)$ and $B_{r_2}(0)$.

Notice that $\sup_{z\in B_{r_1}(0)}|g(z)| \le \sup_{z\in B_{r_2}(0)}|g(z)|$

since $B_{r_1}(0)\subset B_{r_2}(0)$.

Moreover, by max. principle,

$$
\begin{aligned}
f(r_1) &= \sup_{|z|=r_1}|g(z)| \\
&= \sup_{|z|\le r_1}|g(z)| \\
&\le \sup_{z\in B_{r_2}(0)}|g(z)| \\
&= \sup_{|z|=r_2}|g(z)| \\
&= f(r_2)
\end{aligned}
$$

Hence f is increasing.

6) Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$.
Prove that if $|p(z)| \le 1 \quad \forall z \in C(1,0)$, then $p(z) = z^n$.
(Hint: Consider $q(z) = z^n p(\frac{1}{z})$)

Ans: Let $q(z) = z^n p(\frac{1}{z})$

$$= z^n \left(\frac{1}{z^n} + \frac{a_{n-1}}{z^{n-1}} + \cdots + \frac{a_1}{z} + a_0 \right)$$

$$= a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + 1$$

If $|p(z)| \le 1 \quad \forall z \in C(1,0)$,
then $|q(z)| = |z^n p(\frac{1}{z})| = |z|^n |p(\frac{1}{z})| \le 1 \quad \forall z \in C(1,0)$

However, note that $q(0) = 1$
Therefore, $q(z)$ must be a constant function

$\Rightarrow \quad z^n p(\frac{1}{z}) = q(z) = q(0) = 1 \quad \forall z \in \mathbb{C}$

$\Rightarrow \quad a_0 = a_1 = \cdots = a_{n-1} = 0$

$\Rightarrow \quad p(z) = z^n \quad \forall z \in \mathbb{C}$