Some Exercises from §5.3 (cf. P135–136 in Bartle 3rd Ed)

Q5. Let $p(x) = x^4 + 7x^3 - 9$. Then $\exists$ at least two roots of p.
Hint: Note that p is continuous on the interval $I = (-\infty, \infty)$, and that $p(0) = -9 < 0$. Find $x_1 < 0 < x_2$ s.t. $f(x_1)$ and $f(x_2)$ are positive. Apply the Root Th.

Q6. Let $f : [0, 2\pi] \to \mathbb{R}$ be cts s.t. $f(0) = f(2\pi)$. Then $\exists\, c \in [0, \pi]$ s.t. $f(c) = f(c+\pi)$.
Hint: Let $g : [0,\pi] \to \mathbb{R}$ be defined by $g(t) = f(t) - f(t+\pi)$ $\forall\, t \in [0,\pi]$. Then g is cts [?], and $g(0)\, g(\pi) \le 0$.
Apply the Root-Th. [A result from geography: $\exists$ a pair of two antipodal points with the same temperature]

Q11. Let $f : I = [a,b] \to \mathbb{R}$ be cts and $f(a) < 0 < f(b)$.
Let $W := \{t \in I : f(t) < 0\}$. Then, by ...,
$c := \sup W$ exists in $\mathbb{R}$, and $c \in [a,b]$. Show that $f(c) = 0$ (so providing a new proof for the Root Th).
Hint: By def of "sup" (and c) one can find a seq (w_n) in W such that $c - \frac{1}{n} < w_n \le c$ $\forall n$ (so $\lim_n w_n = c$ by the Squeeze Principle) and $f(w_n) < 0$ $\forall n$. Since f is cts on $[a,b]$ and $c \in [a,b]$, f is continuous at c and it follows from the Sequential Criterion for Continuity and the Order-preserving that $f(c) = \lim f(w_n) \le 0$. To show $f(c)$, we suppose on the contrary that $f(c) < 0$. Then $\exists\, \delta > 0$ s.t. $f(\cdot) < 0$ on $V_\delta(c) \cap [a,b]$. Noting $c \ne b$ (as $f(c) < 0 < f(b)$), one has $c < b$; pick $c' \in (c, (c+\delta) \wedge b)$. Then $f(c') < 0$ (and $c' \in [a,b]$) and so $c' \in W$ and $c' > c = \sup W$, contradicting the def of "sup".