# Homework V

Let $f: A \to \mathbb{R}$, and $c \in \mathbb{R}$ be a cluster point w.r.t. $A$.

1* Suppose $(f(x_n))$ converges (in $\mathbb{R}$) whenever $(x_n)$ is a seq in $A \setminus \{c\}$ convergent to $c$. Show that there exists $l \in \mathbb{R}$ such that $(f(x_n))$ converges to $l$ whenever $(x_n)$ is a seq in $A \setminus \{c\}$ convergent to $c$. Hence, by virtue of the definition of limits (for functions), show that $\lim_{x \to c} f(x) = l$.

2* Suppose for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(x')| < \varepsilon \text{ whenever } x, x' \in (A \setminus \{c\}) \cap V_\delta(c).$$

Show that the function $f$ has a limit at $c$.

3* Let $(x_n)$ be a sequence of real numbers. Let

$$s_n = x_1 + \cdots + x_n \quad \text{and} \quad s_n' = |x_1| + \cdots + |x_n| \quad \forall n \in \mathbb{N}$$

(namely $s_n$, $s_n'$ are respectively the $n$-th partial sums of the series $\sum_{n \in \mathbb{N}}^{\infty} x_n$ and $\sum_{n \in \mathbb{N}}^{\infty} |x_n|$). Show

that, if $(s_n')$ converges (to a real number) then $(s_n)$ also converges; this result is often stated as: absolutely summable series is summable.

4* Let $(x_n)$ be a sequence which is <u>not</u> Cauchy.

Show that there is a $\varepsilon > 0$ such that
(i) $\forall N \in \mathbb{N}$ $\exists$ $N' \in \mathbb{N}$ such that $N' > N$ such that $|x_N - x_{N'}| \geq \varepsilon$.
(ii) $\exists$ a subsequence $(x_{n_k})$ such that $|x_{n_k} - x_{n_{k+1}}| \geq \varepsilon$ $\forall k \in \mathbb{N}$.