$\nexists\, f: \mathbb{R} \to \mathbb{R}$ cts at each rational pt but <u>not</u> cts at each irrational.

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(答題不得寫在紅綫外)

<u>Th.</u> Let $f: \mathbb{R} \to \mathbb{R}$ be continuous at each rational. Then $\exists$ an irrational at which f is also continuous.

Proof. Let $\mathbb{Q} = \{r_1, r_2, \cdots\}$ (distinct r_i). Take $n_1 = 1$ and a bounded closed interval I_{n_1} containing r_1 as an interior point (i.e. $r_1 \in I^{\circ}_{n_1}$) such that $|f(x) - f(r_1)| < \frac{1}{2n_1}$ $\forall x \in I_{n_1}$ —

so

$$|f(x) - f(x')| < \frac{1}{n_1} \quad \forall x, x' \in I_{n_1}$$

Pick $n_2 > n_1$ such that $r_{n_2} \in I^{\circ}_{n_1}$ and take a (bounded) closed interval $I_{n_2} \subset I^{\circ}_{n_1}$ s.t

1) containing r_{n_2} as ~~an~~ interior pt : $r_{n_2} \in I^{\circ}_{n_2}$
2) not contain "earlier pts" : $r_i \notin I_{n_2}$ $\forall i < n_2$
3) $|f(x) - f(x')| < \frac{1}{n_2}$ $\forall x, x' \in I_{n_2}$

Inductively, at kth-step, pick $n_k > n_{k-1}$ such that $r_{n_k} \in I^{\circ}_{n_{k-1}}$ and take closed interval $I_{n_k} \subset I^{\circ}_{n_{k-1}}$ such that it

1) contains r_{n_k} as an interior pt : $r_{n_k} \in I^{\circ}_{n_k}$
2) not contain earlier pts : $r_i \notin I_{n_k}$ $\forall i < n_k$

3) $|f(x) - f(x')| < \frac{1}{n_K} \quad \forall\, x, x' \in I_{n_K}$.

Thus we have seq (r_{n_K}) and (I_{n_K}) satisfying 1), 2), and 3) for each K and "strongly nested":

$$I^{\circ}_{n_{K-1}} \supseteq I_{n_K}$$

(so $\bigcap_{K=1}^{\infty} I_K = \bigcap_{K=1}^{\infty} I^{\circ}_K$ is nonempty by the nested interval theorem). Let

$$x_0 \in \bigcap_{K=1}^{\infty} I_K$$

By 2), $x_0 \notin \mathbb{Q}$, while 3) implies that

$$|f(x) - f(x_0)| < \frac{1}{n_K} \quad \forall\, x \in I_{n_K}.$$

Let $\varepsilon > 0$, and let $K \in \mathbb{N}$ be s.t. $\frac{1}{K} < \varepsilon$. Then

$$|f(x) - f(x_0)| < \frac{1}{n_K} \leq \frac{1}{K} < \varepsilon \quad \forall\, x \in I_{n_K} \supseteq I^{\circ}_{n_K}$$

(and $I^{\circ}_{n_K}$ contains a δ-n'd (neighbourhood) of x_0 with some $\delta > 0$).