MATH2040 Linear Algebra II

Tutorial 4

October 6, 2016

1 Examples:

Example 1 (Example of stochastic process) (Textbook: P.288)

Suppose there are two cities named City A and City B, the population in these cities remains constant throughout the years but the is a movement of people between two cities every year. The probabilities that how the citizens may move to another city or not are summarized in the following table:

	Currently living in City A	Currently living in City B
Living in City A next year	0.9	0.02
Living in City B next year	0.1	0.98

(a) Find the probability that someone currently live in City A will move to City B after two years.

(b) Suppose currently there are 600 people live in City A and 600 people live in City B, find the population of City A and City B in the long term, i.e. after many years.

Solution

First, we write the probability matrix $A = \begin{pmatrix} 0.9 & 0.02 \\ 0.1 & 0.98 \end{pmatrix}$ which describes the probabilities of people may move, then $A^2 = \begin{pmatrix} 0.812 & 0.0376 \\ 0.188 & 0.9624 \end{pmatrix}$.

(a) The required probability is just $(A^2)_{21} = 0.188$.

(b) Let $P = \begin{pmatrix} 600 \\ 600 \end{pmatrix}$, then we can find the long term population by considering $A^m P$ as $m \to \infty$.

After determining the eigenvalues and eigenvectors of A, as A has distinct eigenvalues, so A is diagonalizable and

$$\begin{pmatrix} 1 & 0 \\ 0 & 0.88 \end{pmatrix} = D = Q^{-1}AQ = \begin{pmatrix} 1 & 1 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} 0.9 & 0.02 \\ 0.1 & 0.98 \end{pmatrix} \begin{pmatrix} \frac{1}{6} & -\frac{1}{6} \\ \frac{5}{6} & \frac{1}{6} \end{pmatrix}.$$

As

$$\lim_{m\to\infty} A^m = \lim_{m\to\infty} \begin{pmatrix} \frac{1}{6} & -\frac{1}{6} \\ \frac{5}{6} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 1^m & 0 \\ 0 & 0.88^m \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -5 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{5}{6} & \frac{5}{6} \end{pmatrix}.$$

Therefore,

$$\lim_{m\to\infty} A^m P = \begin{pmatrix} 200 \\ 1000 \end{pmatrix},$$

which means after many years, the population of City A and City B will be 200 and 1000 respectively.

Remark: $\lim_{m\to\infty} A^m$ exists only if A satisfies certain conditions.

Example 2

Let T be a diagonalizable linear operator on a finite dimensional vector space V, and let m be any positive integer. Prove that T and T^m are simultaneously diagonalizable.

Definition: Two linear operators T and U on a finite-dimensional vector space V are called simultaneously diagonalizable if there exists an ordered basis β for V such that both $[T]_\beta$ and $[U]_\beta$ are diagonal matrices. Similarly, $A, B \in M_{n\times n}(\mathbb{F})$ are called simultaneously diagonalizable if there exists an invertible matrix $Q \in M_{n\times n}(\mathbb{F})$ such that $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal matrices.

Solution

By the definition of simultaneously diagonalizable, we need to show that there exists an ordered basis β for V such that $[T]_\beta$ and $[U]_\beta$ are both diagonal matrices. In other words, β is eigenbasis for both T and T^m.

WLOG, assume $\dim(V) = n$, and since T is a diagonalizable linear operator, so there exists an eigenbasis $\beta = \{\beta_1, \beta_2, \ldots, \beta_n\}$ such that $[T]_\beta$ ia diagonal. Note,

$$T(\beta_i) = \lambda_i \beta_i \Rightarrow T^2(\beta_i) = \lambda_i T(\beta_i) \Rightarrow T^2(\beta_i) = \lambda_i^2 \beta_i \Rightarrow T^m(\beta_i) = \lambda_i^m \beta_i \quad \forall i = 1, 2, ..., n.$$

Therefore, β_i is also eigenvector of U with corresponding eigenvalue λ_i^m for all $i = 1, ..., n$. So β is an eigenbasis of T^m and $[T^m]_\beta$ is diagonal.

2 Exercises:

Question 1 (Section 5.2 Q7):

For $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$, find an expression for A^n, where n is an arbitrary positive integer.

Question 2 (Section 5.2 Q8):

Suppose that $A \in M_{n\times n}(\mathbb{F})$ has two distinct eigenvalues, λ_1 and λ_2, and that $\dim(E_{\lambda_1}) = n - 1$. Prove that A is diagonalizable.

Question 3 (Section 5.2 Q17):

(a) Prove that if T and U are simultaneously diagonalizable linear operators on a finite dimensional vector space V, then the matrices $[T]_\gamma$ and $[U]_\gamma$ are simultaneously diagonalizable for any ordered basis γ.

(b) Prove that if A and B are simultaneously diagonalizable matrices, then L_A and L_B are simultaneously diagonalizable linear operators.

Question 4 (Section 5.2 Q18):

(a) Prove that if T and U are simultaneously diagonalizable linear operators, then T and U commute (i.e. $TU = UT$).

(b) Prove that if A and B are simultaneously diagonalizable matrices, then A and B commute.

Solution

(Please refer to the practice problem set 4.)