Lecture 19 (2): Unitary and orthogonal operators

Recall that: T^* shares similar properties as complex conjugate:

e.g. $T^{**} = T$ $\quad (\bar{\bar{a}} = a)$

$I^* = I$ $\quad (\bar{1} = 1)$

$(cT)^* = \bar{c}T^*$ $\quad (\overline{ab} = \bar{a}\,\bar{b})$

Interesting question in complex analysis: study λ such that $|\lambda|^2 = \lambda\bar{\lambda} = 1$ (Complex number with unit length)

Goal: • Study T such that $TT^* = T^*T = I$

• Study its properties.

Definition: Let T be a linear operator on a fin-dim inner product space V over $\mathbb{F}$. If $\|T(\vec{x})\| = \|\vec{x}\|$ for all $\vec{x} \in V$, we say T is a unitary operator if $\mathbb{F} = \mathbb{C}$, and we say T is an orthogonal operator if $\mathbb{F} = \mathbb{R}$.

Remark: • For infinite dim V, if $\|T(\vec{x})\| = \|\vec{x}\|$ for all $\vec{x} \in V$, then T is called an isometry.

Example: Let $h \in C([0, 2\pi])$ = collection of all complex-valued continuous function on $[0, 2\pi]$.

be defined as $h(x) = e^{iRx}$ $(R \in \mathbb{R})$.

Define $T(f) = e^{iRx} f(x)$. Then, T is onto.

Clearly, $\|Tf\|^2 = \|hf\|^2 = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{h(t)}_{e^{iRt}} f(t)\, \underbrace{\overline{h(t)}}_{e^{-iRt}}\, \overline{f(t)}\, dt = \|f\|^2$

$\therefore$ T is an unitary operator.

Lemma: Let U be a self-adjoint operator on a fin-dim inner product space V. If $\langle \vec{x}, U(\vec{x}) \rangle = 0$ for all $\vec{x} \in V$, then $U = T_0$ $\left(T_0(\vec{x}) = 0 \text{ for } \forall \vec{x} \right)$

Proof: Since U is self-adjoint, there exists an orthonormal basis β of eigenvectors of T. Let $\vec{x} \in \beta$.

Then, $U(\vec{x}) = \lambda \vec{x}$ for some λ.

Thus, $0 = \langle \vec{x}, U(\vec{x}) \rangle = \langle \vec{x}, \lambda \vec{x} \rangle = \bar{\lambda} \underbrace{\langle \vec{x}, \vec{x} \rangle}_{\neq 0}$

and so $\bar{\lambda} = 0$.

Hence, $U(\vec{x}) = \vec{0}$ for all basis element $\vec{x} \in \beta$.

We conclude that $U(\vec{x}) = \vec{0}$ for all $\vec{x} \in V$.

Theorem: Let T be a lin operator on a fin-dim inner product space V. The following are equivalent:

(a) $TT^* = T^*T = I$

(b) $\langle T(\vec{x}), T(\vec{y}) \rangle = \langle \vec{x}, \vec{y} \rangle$ for all $\vec{x}, \vec{y} \in V$.

(c) If β is an orthonormal basis for V, then $T(\beta)$ is an orthonormal basis for V.

(If $\beta = \{ \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \}$, then $T(\beta) = \{ T(\vec{v}_1), T(\vec{v}_2), \dots, T(\vec{v}_n) \}$.

(d) There exists an orthonormal basis for V such that $T(\beta)$ is an orthonormal basis for V.

(e) $\| T(\vec{x}) \| = \| \vec{x} \|$ for all $\vec{x} \in V$.

Proof: (a) $\Rightarrow$ (b): Let $\vec{x}, \vec{y} \in V$.

Then: $\langle \vec{x}, \vec{y} \rangle = \langle T^* T(\vec{x}), \vec{y} \rangle = \langle T(\vec{x}), T(\vec{y}) \rangle$.

(b) $\Rightarrow$ (c): Let $\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ = orthonormal basis for V. Consider $T(\beta) = \{T(\vec{v}_1), T(\vec{v}_2), \dots, T(\vec{v}_n)\}$.

Then, $\langle T(\vec{v}_i), T(\vec{v}_j) \rangle = \langle \vec{v}_i, \vec{v}_j \rangle = \delta_{ij}$

$\therefore$ $T(\beta)$ is an orthonormal set and so

$T(\beta)$ is an orthonormal basis for V.

(c) $\Rightarrow$ (d): Obvious.

(d) $\Rightarrow$ (e): Let $\vec{x} \in V$ and $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$.

Let $\vec{x} = \sum_{i=1}^{n} a_i \vec{v}_i$ for some $a_i \in \mathbb{F}$.

$$\text{So, } \|\vec{x}\|^2 = \left\langle \sum_{i=1}^{n} a_i \vec{v}_i, \sum_{j=1}^{n} a_j \vec{v}_j \right\rangle$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a_j} \underbrace{\langle \vec{v}_i, \vec{v}_j \rangle}_{\delta_{ij}} = \sum_{i=1}^{n} |a_i|^2$$

Apply the same technique,

$$T(\vec{x}) = \sum_{i=1}^{n} a_i T(\vec{v}_i)$$

$$\therefore \|T(\vec{x})\|^2 = \left\langle \sum_{i=1}^{n} a_i T(\vec{v}_i), \sum_{j=1}^{n} a_j T(\vec{v}_j) \right\rangle$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a_j} \underbrace{\langle T(\vec{v}_i), T(\vec{v}_j) \rangle}_{\delta_{ij}}$$

$$= \sum_{i=1}^{n} |a_i|^2$$

Hence, $\|T(\vec{x})\| = \|\vec{x}\|$.

(e) $\Rightarrow$ (a): For $\forall \vec{x} \in V$, we have:

$\langle \vec{x}, \vec{x} \rangle = \|\vec{x}\|^2 = \|T(\vec{x})\|^2 = \langle T(\vec{x}), T(\vec{x}) \rangle = \langle \vec{x}, T^*T(\vec{x}) \rangle$

So, $\langle \vec{x}, \underbrace{(I - T^*T)}_{U}(\vec{x}) \rangle = 0$ for $\forall \vec{x} \in V$.

Let $U = I - T^*T$. Then, U is self-adjoint.

Hence, $\langle \vec{x}, U(\vec{x}) \rangle = 0$ for $\forall \vec{x} \in V$.

By lemma, $U = I - T^*T = T_0$ = zero transformation.

$\therefore$ $T^*T = I$.

Since V is finite-dimensional, $T^*T = TT^* = I$.