Lecture 17: Adjoint of a linear operator

Last time: We consider $V^* := \{f : V \to \mathbb{F} : f \text{ is linear}\}$

V^* is often called the dual space of V.

Theorem 1: Let V = fin-dim inner product space. $T: V \to \mathbb{F}$ be linear. Then, $\exists$ unique function $T^* : V \to V$ such that $\langle T(\vec{x}), \vec{y} \rangle = \langle \vec{x}, T^*(\vec{y}) \rangle$ and T^* is linear.

(T^* is called the adjoint of T)

Proof: Let $\vec{y} \in V$. Define $g : V \to \mathbb{F}$ by $g(x) = \langle T(\vec{x}), \vec{y} \rangle$ for $\forall \vec{x} \in V$

g is linear: $g(c\vec{x}_1 + \vec{x}_2) = \langle T(c\vec{x}_1 + \vec{x}_2), \vec{y} \rangle = \langle cT(\vec{x}_1) + T(\vec{x}_2), \vec{y} \rangle$

$$= c\langle T(\vec{x}_1), \vec{y} \rangle + \langle T(\vec{x}_2), \vec{y} \rangle = cg(\vec{x}_1) + g(\vec{x}_2)$$

So, $\exists$ unique $\vec{y}\,' \in V$ such that $g(\vec{x}) = \langle T(\vec{x}), y \rangle = \langle x, \vec{y}\,' \rangle$

Define : $T^*(\vec{y}) = \vec{y}\,'$.

Now, we show T^* is linear. Let $\vec{y}_1, \vec{y}_2 \in V$, $c \in \mathbb{F}$. For any $\vec{x} \in V$, we have: $\langle \vec{x}, T^*(c\vec{y}_1 + \vec{y}_2) \rangle = \langle T(\vec{x}), c\vec{y}_1 + \vec{y}_2 \rangle = \bar{c}\langle T(\vec{x}), \vec{y}_1 \rangle + \langle T(\vec{x}), \vec{y}_2 \rangle$

$$= \bar{c}\langle \vec{x}, T^*(\vec{y}_1) \rangle + \langle \vec{x}, T^*(\vec{y}_2) \rangle$$

$$= \langle \vec{x}, cT^*(\vec{y}_1) \rangle + \langle \vec{x}, T^*(\vec{y}_2) \rangle$$

$$= \langle \vec{x}, cT^*(\vec{y}_1) + T^*(\vec{y}_2) \rangle$$

$\vec{x}$ is arbitrary $\Rightarrow T^*(c\vec{y}_1 + \vec{y}_2) = cT^*(\vec{y}_1) + T^*(\vec{y}_2)$

To show T^* is unique, suppose $U : V \to V$ is linear and $\langle T(\vec{x}), \vec{y} \rangle = \langle \vec{x}, U(\vec{y}) \rangle$ for $\forall \vec{x}, \vec{y} \in V$

$\Rightarrow \langle \vec{x}, U(\vec{y}) \rangle = \langle \vec{x}, T^*(\vec{y}) \rangle$ for $\forall \vec{x}, \vec{y} \in V$

$\Rightarrow U(\vec{y}) = T^*(\vec{y})$ for $\forall \vec{y} \Rightarrow U = T^*$.

Remark: • T^* is called the adjoint of T.

• Not only $\langle Tx, y\rangle = \langle x, T^* y\rangle$.

$\langle x, Ty\rangle = \overline{\langle Ty, x\rangle} = \overline{\langle y, T^* x\rangle} = \langle T^* x, y\rangle$

(When shifting location of T, add "*")

Theorem 2: Let V = fin-dim inner product space. β = orthonormal basis for V. Let $T: V \to V$ lin.

Then: $[T^*]_\beta = ([T]_\beta)^*$ (Adjoint of transformation related to adjoint of matrix)

Proof: Let $A = [T]_\beta$. $B = [T^*]_\beta$ where $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$ = o.n. basis.

Then: $B_{ij} = \langle T^*(\vec{v}_j), \vec{v}_i\rangle = \overline{\langle \vec{v}_i, T^*(\vec{v}_j)\rangle} = \overline{\langle T(\vec{v}_i), \vec{v}_j\rangle}$

$= \overline{A_{ji}} = (A^*)_{ij}$

Hence, $B^* = A$.

Corollary 1: Let $A \in M_{n\times n}(\mathbb{F})$. Then: $L_{A^*} = (L_A)^*$.

Recall: $L_A : \mathbb{F}^n \to \mathbb{F}^n$ where $L_A(\underset{\in \mathbb{F}^n}{\vec{x}}) = A\vec{x}$.

Proof: β = standard o.n. basis for $\mathbb{F}^n$.

Then: $[(L_A)^*]_\beta = ([L_A]_\beta)^* = A^* = [L_{A^*}]_\beta$

$\therefore (L_A)^* = L_{A^*}$.

Example 1: (How to compute adjoint of lin operator)

Let $T: \mathbb{C}^3 \to \mathbb{C}^3$ by $T(a_1, a_2, a_3) = (ia_1 + 2a_2, 3ia_2 + 5a_3, 7ia_1 + 8ia_3)$

Then: $[T]_\beta = \begin{pmatrix} i & 2 & 0 \\ 0 & 3i & 5 \\ 7i & 0 & 8i \end{pmatrix}$.

Recall: $[T^*]_\beta = [T]_\beta^* = \begin{pmatrix} i & 2 & 0 \\ 0 & 3i & 5 \\ 7i & 0 & i \end{pmatrix}^* = \begin{pmatrix} -i & 0 & -7i \\ 2 & -3i & 0 \\ 0 & 5 & -i \end{pmatrix}$

Thus, $T^*(a_1, a_2, a_3) = (-ia_1 - 7i a_3, 2a_1 - 3ia_2, 5a_2 - ia_3)$

Remark: Using matrix adjoint (easy computation) to find lin. operator adjoint.

Theorem 3: Let V = inner product space. Let $T, U: V \to V$ be lin. op. Then:

(a) $(T+U)^* = T^* + U^*$; (b) $(cT)^* = \bar{c}T^*$ for $\forall c \in \mathbb{F}$.

(c) $(TU)^* = U^*T^*$ (d) $T^{**} = T$ (e) $I^* = I$.

Proof: For any $x, y \in V$,

(a) $\langle x, (T+U)^* y\rangle = \langle (T+U)(x), y\rangle = \langle T(x) + U(x), y\rangle$

$= \langle T(x), y\rangle + \langle U(x), y\rangle = \langle x, T^*y\rangle + \langle x, U^*y\rangle$

$= \langle x, (T^* + U^*)y\rangle$

$\therefore (T+U)^* = T^* + U^*$

(b) $\langle x, (cT)^*(y)\rangle = \langle (cT)(x), y\rangle = \langle cT(x), y\rangle = c\langle T(x), y\rangle$

$= \langle T(x), \bar{c}y\rangle = \langle x, T^*(\bar{c}y)\rangle$

$= \langle x, (\bar{c}T^*)(y)\rangle$

$\therefore (cT)^* = \bar{c}T^*$

(c) $\langle x, (TU)^*y\rangle = \langle (TU)(x), y\rangle = \langle U(x), T^*y\rangle$

$= \langle x, U^*T^*y\rangle$ $\therefore (TU)^* = U^*T^*$.

(d) $\langle x, T(y)\rangle = \langle T^*x, y\rangle = \langle x, T^{**}y\rangle$ $\therefore T = T^{**}$.

<u>Corollary</u>: $A, B \in M_{n\times n}(\mathbb{F})$. Then:

(a) $(A+B)^* = A^* + B^*$

(b) $(cA)^* = \bar{c} A^*$

(c) $(AB)^* = B^* A^*$

(d) $A^{**} = A$

(e) $I^* = I$

<u>Proof</u>: $L_{(AB)^*} = (L_{AB})^* = (L_A L_B)^* = (L_B)^* (L_A)^* = L_{B^*} L_{A^*} = L_{B^* A^*}$

$\therefore (AB)^* = B^* A^*$

Normal and self-adjoint operators

<u>Recall</u>: T diagonalizable $\Leftrightarrow$ V has basis of eigenvectors.

If V has basis of eigenvectors + orthonormal, then computation much simpler.

<u>Goal</u>: Orthonormal basis of eigenvectors.

<u>Lemma</u>: $T: V \to V$ (fin-dim inner product space)

If T has an eigenvector, then T^* also has eigenvector.

<u>Proof</u>: $\vec{v}$ = eigenvector of T w/ eigenvalue λ.

For any $x \in V$, $0 = \langle \vec{0}, x \rangle = \langle (T - \lambda I)(v), x \rangle = \langle v, (T^* - \bar{\lambda} I)(x) \rangle$

$\therefore v$ is orthogonal to $R(T^* - \bar{\lambda} I)$ and so $R(T^* - \bar{\lambda} I)^{\perp} \neq \{\vec{0}\}$.

Thus, $R(T^* - \bar{\lambda} I) \neq V$. Hence, $N(T^* - \bar{\lambda} I) \neq \{\vec{0}\}$

Any non-zero vector in $N(T^* - \bar{\lambda} I)$ is an eigenvector of T^* w/ eigenvale $\bar{\lambda}$.

<u>Theorem 4</u>: (Schur) Let T = lin op. of fin-dim V. Suppose char. poly. of T splits, then $\exists$ orthonormal basis β of V such that $[T]_\beta$ is upper triangular.

Proof: By M.I. on the dim n of V.

When $n=1$, obvious since $[T]_\beta = 1\times 1$ matrix.

Suppose the thm is true for lin op. on $(n-1)$-dim inner product space whose char. poly. splits.

By lemma, we can assume T^* has unit eigenvector z. (Say $T^*z=\lambda z$)

Let $W = \text{span}\{z\}$. We prove $W^\perp$ is T-invariant.

(If $W^\perp$ is T-invariant, we can consider $T|_{W^\perp} : \underset{\uparrow \text{ n-1 dim}}{W^\perp} \to W^\perp$ and apply induction assumption)

Let $y \in W^\perp$ and $x = cz \in W$.

Then: $\langle T(y), x\rangle = \langle y, T^* x\rangle = \langle y, T^*(cz)\rangle = \langle y, c\lambda z\rangle$

$$= \bar{c}\bar{\lambda}\langle y, z\rangle = 0$$

$\therefore$ $T(y) \in W^\perp$ and $W^\perp$ is T-invariant.

Now, $\dim(W^\perp) = n-1$. The char poly of $T|_{W^\perp}$ divides char poly of T. Hence, char poly of $T|_{W^\perp}$ splits. By induction, $\exists$ orthonormal basis γ of $W^\perp$ such that $[T_{W^\perp}]_\gamma$ is upper triangular. Let $\beta = \gamma \cup \{z\}$. Then β is an orthonormal basis for V such that $[T]_\beta = \begin{pmatrix} [T]_\gamma & \vdots \\ O & \lambda \end{pmatrix}$ is upper triangular.

<u>Definition</u>: V = inner product space. Let $T: V \to V$ lin op on V. We say T is <u>normal</u> if $TT^* = T^*T$. A matrix $A \in M_{n\times n}(\mathbb{F})$ is normal if $AA^* = A^*A$.

<u>Motivation</u>: Our goal is to find orthonormal basis of eigenvectors.

Suppose such basis β exist.

Then: $[T]_\beta$ = diagonal. So, $[T^*]_\beta = [T]_\beta^*$ is also diagonal.

Thus, $[T]_\beta [T]_\beta^* = [T]_\beta^* [T]_\beta$

$$\underbrace{[T]_\beta [T^*]_\beta}_{=[TT^*]_\beta} \qquad \underbrace{[T^*]_\beta [T]_\beta}_{=[T^*T]_\beta} \qquad \therefore \quad TT^* = T^*T.$$

Hence, if such basis exist, then T must be normal.

<u>Example</u>: $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Then $AA^* = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Also, $A^*A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \therefore AA^* = A^*A$ ← A is normal.

Since A is the matrix rep of rotation transformation T,

rotational transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is <u>normal</u>.