# Lecture 14: Orthonormal basis

**Recall**: $S = \{\vec{v}_1, \dots, \vec{v}_n\}$ = orthogonal if $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ if $i \neq j$.

$S = \{\vec{v}_1, \dots, \vec{v}_n\}$ = orthonormal if $S$ is orthogonal and $\langle \vec{v}_i, \vec{v}_i \rangle = 1$ for $\forall i$

If $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ = orthogonal basis of $V$, then for $\forall \vec{v} \in V$,

$$\vec{v} = \sum_{i=1}^{n} \frac{\langle \vec{v}, \vec{v}_i \rangle}{\langle \vec{v}_i, \vec{v}_i \rangle} \vec{v}_i$$

coefficient can be easily found

If $\beta$ is orthonormal, then: $\vec{v} = \sum_{i=1}^{n} \langle \vec{v}, \vec{v}_i \rangle \vec{v}_i$ (Even easier)

**Goal**: Find orthonormal basis!

Basis $\Rightarrow$ independent.

**Theorem 1**: Orthogonal subset (without $\vec{0}$) is linearly independent.

**Pf**: Let $\vec{0} = \sum_{i=1}^{k} a_i \vec{v}_i$ where $\vec{v}_1, \dots, \vec{v}_k \in S$.

Then: $\langle \vec{0}, \vec{v}_j \rangle = 0 = \langle \sum_{i=1}^{k} a_i \vec{v}_i, \vec{v}_j \rangle = \sum_{i=1}^{k} a_i \langle \vec{v}_i, \vec{v}_j \rangle$

$= a_j \langle \vec{v}_j, \vec{v}_j \rangle$

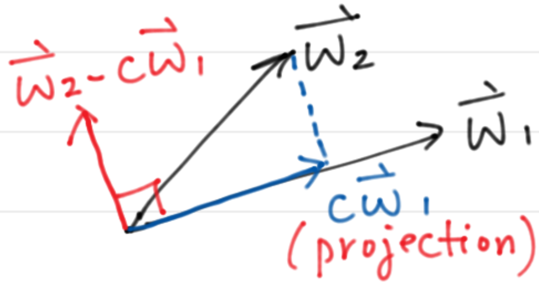
$\therefore a_j = 0$ for $\forall j$.

Let $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$. How to modify $\beta$ to get orthonormal basis $\gamma$?

Consider $\mathbb{R}^2$ case first!

**Observation:** Consider $V=\mathbb{R}^2$. Let $V=\text{span}\underbrace{\{\vec{w}_1,\vec{w}_2\}}_{\text{lin. ind.}}$

Geometrically,




**Goal:** Find $c\in\mathbb{R} \ni \vec{w}_2 - c\vec{w}_1 \perp \vec{w}_1$

To find $c$: $0=\langle \vec{w}_2 - c\vec{w}_1, \vec{w}_1\rangle = \langle \vec{w}_2, \vec{w}_1\rangle - c\langle \vec{w}_1, \vec{w}_1\rangle$

$$\Rightarrow c = \frac{\langle \vec{w}_2, \vec{w}_1\rangle}{\|\vec{w}_1\|^2}$$

We let $\vec{v}_1 = \vec{w}_1$ and $\vec{v}_2 = \vec{w}_2 - \overbrace{\frac{\langle \vec{w}_2, \vec{w}_1\rangle}{\|\vec{w}_1\|^2}}^{c}\vec{w}_1$

Then: $\{\vec{v}_1, \vec{v}_2\}$ = orthogonal basis.

and $\left\{\frac{\vec{v}_1}{\|\vec{v}_1\|}, \frac{\vec{v}_2}{\|\vec{v}_2\|}\right\}$ = orthonormal basis (normalization)

This idea can be extended to general vector space (fin. dim)

**Theorem 2:** Let $V$ = inner product space. $S=\{\vec{w}_1,\dots,\vec{w}_n\}$ = lin. ind. subset. Define $S'=\{\vec{v}_1,\dots,\vec{v}_n\}$ by:

① $\vec{v}_1 = \vec{w}_1$

② $\vec{v}_k = \vec{w}_k - \sum_{j=1}^{k-1} \frac{\langle \vec{w}_k, \vec{v}_j\rangle}{\|\vec{v}_j\|^2}\vec{v}_j$, $2\le k\le n$

Then: $S'$ = non-zero orthogonal set and $\text{span}(S) = \text{span}(S')$

**Remark:** If $S$ is a basis of $V$, then:

$S'$ = orthogonal $\Rightarrow$ lin. ind. and $\text{span}(S') = \text{span}(S) = V$. $\therefore S'$ = orthogonal basis.

Proof: By M.I. on $n$.

For $k=1,2,\dots,n$. Define: $S_k = \{\vec{w}_1, \dots, \vec{w}_k\}$.

For $n=1$, the thm is trivially true.

Assume the thm is true for $n=k-1$.

When $n=k$, we prove that $S_k' = \{\vec{v}_1, \dots, \vec{v}_{k-1}, \vec{v}_k\}$ satisfies the thm. Let $S'_{k-1} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{k-1}\}$.

Recall: $\vec{v}_k = \vec{w}_k - \sum_{j=1}^{k-1} \frac{\langle \vec{w}_k, \vec{v}_j \rangle}{\|\vec{v}_j\|^2} \vec{v}_j$ ——— $(*)$

If $\vec{v}_k = \vec{0}$, then $\vec{w}_k \in \mathrm{Span}(S'_{k-1}) = \mathrm{Span}(S_{k-1})$ (induction hypothesis)

Contradicting that $S_k$ is lin. ind. $\therefore \vec{v}_k \neq \vec{0}$.

For $1 \le i \le k-1$, $\langle \vec{v}_k, \vec{v}_i \rangle = \langle \vec{w}_k, \vec{v}_i \rangle - \sum_{j=1}^{k-1} \frac{\langle \vec{w}_k, \vec{v}_j \rangle}{\|\vec{v}_j\|^2} \langle \vec{v}_j, \vec{v}_i \rangle$

(using $*$)

(By orthogonality of $S'_{k-1}$) $= \langle \vec{w}_k, \vec{v}_i \rangle - \frac{\langle \vec{w}_k, \vec{v}_i \rangle}{\cancel{\|\vec{v}_i\|^2}} \cancel{\langle \vec{v}_i, \vec{v}_i \rangle}$

$= \vec{0}$

Hence: $S_k'$ is <u>orthogonal</u> set of <u>non-zero</u> vectors.

By $(*)$, we know: $\mathrm{Span}(S_k') \subseteq \mathrm{Span}(S_k)$ because:

$\vec{v}_i$ = lin. comb. of $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_i$.

Also, $S_k'$ = orthogonal $\Rightarrow S_k'$ = lin. ind.

$\therefore \dim(\mathrm{span}(S_k')) = \dim(\mathrm{span}(S_k)) = k$

$\therefore \mathrm{span}(S_k') = \mathrm{span}(S_k)$.

**Remark:** Such process of finding orthogonal set is called the <u>Gram-Schmidt process</u>.

**Example 1:** Consider $\mathbb{R}^3$. Let $\vec{w}_1 = (1,0,1)$, $\vec{w}_2 = (1,1,0)$, $\vec{w}_3 = (0,0,1)$

Then: $\beta = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is lin. ind. and hence form a basis of $\mathbb{R}^3$.

**Goal:** Use Gram-Schmidt process to get orthogonal basis.

Take $\vec{v}_1 = \vec{w}_1 = (1,0,1)$

Compute: $\vec{v}_2 = \vec{w}_2 - \frac{\langle \vec{w}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = (1,1,0) - \frac{1}{2}(1,0,1)$
$= (\frac{1}{2}, 1, -\frac{1}{2})$

Similarly, compute: $\vec{v}_3 = \vec{w}_3 - \frac{\langle \vec{w}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{w}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$
$= (-\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

Then: $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set.

Again, if we want to get an orthonormal set, we do normalization:

$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{(1,0,1)}{\sqrt{2}}$; $\vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{(\frac{1}{2}, 1, -\frac{1}{2})}{\sqrt{\frac{3}{2}}}$; $\vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \frac{(-\frac{1}{3}, \frac{1}{3}, \frac{1}{3})}{\sqrt{\frac{1}{3}}}$

Then: $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ = orthonormal set.

**Remark:** • Need to remember:

$$\vec{v}_k = \vec{w}_k - \frac{\langle \vec{w}_k, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \cdots - \frac{\langle \vec{w}_k - \vec{v}_{k-1} \rangle}{\|\vec{v}_{k-1}\|^2}\vec{v}_{k-1}$$

• Every vector space $V$ has an orthonormal basis!

**Theorem 3:** Every vector space $V$ has an orthonormal basis.

**Proof:** Let $\beta_0 =$ basis of $V$. Do G-S process to get orthogonal set $\beta'$ of non-zero vectors such that $\text{span}(\beta') = \text{span}(\beta_0) = V$.

$\therefore \beta' =$ orthogonal basis. Normalize $\beta'$ to get an orthonormal basis.

**Corollary 1:** Let $V =$ fin-dim inner product space with orthonormal basis $\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. Let $T =$ lin. operator on $V$. and let $A = [T]_\beta$. Then: $A_{ij} = \langle T(\vec{v}_j), \vec{v}_i \rangle$ for $\forall i, j$.

**Proof:** Recall: $A = \begin{pmatrix} | & & | & \\ [T(\vec{v}_1)]_\beta & \cdots & [T(\vec{v}_j)]_\beta & \cdots \\ | & & | & \end{pmatrix}$

In other words, $T(\vec{v}_j) = A_{1j}\vec{v}_1 + A_{2j}\vec{v}_2 + \dots + A_{ij}\vec{v}_i + \dots + A_{nj}\vec{v}_n$

So, $\langle T(\vec{v}_j), \vec{v}_i \rangle = \sum_{k=1}^{n} A_{kj} \langle \vec{v}_k, \vec{v}_i \rangle = A_{ij} \langle \vec{v}_i, \vec{v}_i \rangle = A_{ij}$.

**Theorem 4:** Let $S = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n\} =$ orthogonal set. Then: using G-S process, $S' = S$.

**Proof:** By M.I on $n$. Sketch of proof:

$\vec{v}_1 = \vec{w}_1$ okay.

$$\vec{v}_2 = \vec{w}_2 - \underbrace{\frac{\langle \vec{w}_2, \overset{= \vec{w}_1}{\vec{v}_1} \rangle}{\|\vec{v}_1\|^2}}_{= 0} \vec{v}_1 = \vec{w}_2$$

$$\vec{v}_3 = \vec{w}_3 - \underbrace{\frac{\langle \vec{w}_3, \overset{= \vec{w}_1}{\vec{v}_1} \rangle}{\|\vec{v}_1\|^2}}_{= 0} \vec{v}_1 - \underbrace{\frac{\langle \vec{w}_3, \overset{= \vec{w}_2}{\vec{v}_2} \rangle}{\|\vec{v}_2\|^2}}_{= 0} \vec{v}_2 = \vec{w}_3$$

$\vdots$