THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2020A (FIRST TERM, 2016 - 2017)
ADVANCED CALCULUS II TERM TEST II

Time: 45 minutes Total Marks: 20

Each question worth 5 marks. Explain your answers.

(1) Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be the map defined by $\varphi(x,y) = (x^2+y, x-y^2)$ and let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x,y) = x$. Show that φ is one to one on $[0,1]^2$ and find $\int_{\varphi([0,1]^2)} f$.

Solution: If $x^2 + y = x'^2 + y'$ and $x - y^2 = x' - y'^2$, then

$$\begin{aligned} (x-x')(x+x') &= y'-y \\ x - x' &= (y-y')(y+y'). \end{aligned} \tag{0.1}$$

By the second equation, $x = x'$ if $y = y'$. If $y < y'$, then the second equation implies $x - x' < 0$ and the first equation implies that $x - x' > 0$. So $(x,y) = (x',y')$.

$$D\varphi = \begin{pmatrix} 2x & 1 \\ 1 & -2y \end{pmatrix}$$

By change of variables formula and Fubini's theorem,

$$\begin{aligned} &\int_{\varphi([0,1]^2)} f = \int_{[0,1]^2} f(\varphi)|\det(D\varphi)| \\ &= \int_0^1 \int_0^1 (x^2+y)(4xy+1)dxdy \\ &= \int_0^1 \int_0^1 (4x^3y + x^2 + 4xy^2 + y)dxdy \\ &= \int_0^1 \left(y + \frac{1}{3} + 2y^2 + y \right) dy \\ &= 2 \end{aligned}$$

(2) Let U be the subset of $\mathbb{R}^3$ defined by

$$U = \{(x,y,z) | x^2 + y^2 \leq 1 + z \text{ and } 0 \leq z \leq 4\}.$$

Find the volume of U.

Solution: Let $\varphi : (0,1)\times(0,2\pi)\times(0,4)\to U$ be the map defined by

$$\varphi(r,\theta,z)=(r\sqrt{1+z}\cos\theta, r\sqrt{1+z}\sin\theta, z).$$

The derivative $D\varphi$ is given by

$$D\varphi=\begin{pmatrix} \sqrt{1+z}\cos\theta & -r\sqrt{1+z}\sin\theta & \frac{r\cos\theta}{2\sqrt{1+z}} \\ \sqrt{1+z}\sin\theta & r\sqrt{1+z}\cos\theta & \frac{r\cos\theta}{2\sqrt{1+z}} \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, by the change of variable formula and Fubini's theorem, the volume of U is given by

$$\int_0^1\int_0^{2\pi}\int_0^4 r(1+z)\,dz\,d\theta\,dr=12\pi$$

(3) Recall that if $z=x+iy$ is a complex number, then $e^z=e^x(\cos(y)+i\sin(y))$. Let

$$M_1=\{(z,e^z+e^{-z})\in\mathbb{C}^2|z\in\mathbb{C}\}$$

and let

$$M_2=\{(x+iy,w)\in\mathbb{C}^2|-1\le x\le 1 \text{ and } -1\le y\le 1\}.$$

Find the 2-dimensional volume of the surface $M_1\cap M_2$.

Solution: Let $\varphi:(-1,1)\times(-1,1)\to M_1\cap M_2$ be defined by

$$\varphi(x,y)=(x,y,(e^x+e^{-x})\cos(y),(e^x-e^{-x})\sin(y)).$$

Its derivative is given by

$$D\varphi=\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ (e^x-e^{-x})\cos(y) & -(e^x+e^{-x})\sin(y) \\ (e^x+e^{-x})\sin(y) & (e^x-e^{-x})\cos(y) \end{pmatrix}$$

It follows that

$$D\varphi^T D\varphi=(1+(e^x-e^{-x})^2\cos^2(y)+(e^x+e^{-x})^2\sin^2(y))I$$
$$=(1+e^{2x}+e^{-2x}-2\cos(2y))I.$$

Therefore, the surface area of $M_1\cap M_2$ is given by

$$\int_{-1}^1\int_{-1}^1 1+e^{2x}+e^{-2x}-2\cos(2y)dxdy$$
$$=4+2\int_{-1}^1 e^{2x}+e^{-2x}dx-4\int_{-1}^1\cos(2y)dy$$
$$=4+2e^2-2e^{-2}-4\sin(2)$$

(4) Let $B_R^{n+1} := \{x \in \mathbb{R}^{n+1} \| x| \leq R\}$ be the $n+1$-dimensional ball of radius R centered at the origin and let $S_R^n := \{x \in \mathbb{R}^{n+1} \| x| = R\}$ be the n-dimensional sphere of radius R centered at the origin. Show that $\mathbf{vol}_n(S_R^n) = \frac{n+1}{R}\mathbf{vol}_{n+1}(B_R^{n+1})$, where $\mathbf{vol}_k$ denotes the k-dimensional volume.

Solution: Let $\varphi : B_1^n \times (0,R) \to B_R^{n+1} \cap \{x_{n+1} \geq 0\}$ be the map defined by

$$\varphi(x,r) = (rx, r\sqrt{1-|x|^2}).$$

It follows that

$$D\varphi = \begin{pmatrix} rI & x \\ -\frac{rx^T}{\sqrt{1-|x|^2}} & \sqrt{1-|x|^2} \end{pmatrix}$$

and

$$\det(D\varphi) = \frac{r^n}{\sqrt{1-|x|^2}} \det \begin{pmatrix} I & x \\ -x^T & 1-|x|^2 \end{pmatrix}$$

$$= \frac{r^n}{\sqrt{1-|x|^2}} \det \begin{pmatrix} I & x \\ 0 & 1 \end{pmatrix} = \frac{r^n}{\sqrt{1-|x|^2}}$$

Then $\mathbf{vol}_{n+1}(B_R^{n+1})$ is given by

$$\mathbf{vol}_{n+1}(B_R^{n+1}) = 2\int_0^R \int_{B_1^n} \frac{r^n}{\sqrt{1-|x|^2}} dx dr = 2\frac{R^{n+1}}{n+1} \int_{B_1^n} \frac{1}{\sqrt{1-|x|^2}} dx$$

Let $\psi(x) = \varphi(x,R)$. Then

$$D\psi^T D\psi = R^2 I + \frac{R^2}{1-|x|^2} xx^T.$$

It follows that

$$\det(D\psi^T D\psi) = R^{2n} \det\left(I + \frac{1}{1-|x|^2} xx^T\right) = \frac{R^{2n}}{1-|x|^2}.$$

Therefore,

$$\mathbf{vol}_n(S_R^n) = 2R^n \int_{B_1^n} \frac{1}{\sqrt{1-|x|^2}} = \frac{n+1}{R} \mathbf{vol}_{n+1}(B_R^{n+1})$$