THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2020A (FIRST TERM, 2016 - 2017)
ADVANCED CALCULUS II TERM TEST I SOLUTION

(1) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x,y) = \frac{\cos x}{x}$. Find the integral $\int_0^\pi \left(\int_y^\pi f(x,y)dx\right) dy$.

Solution: $\int_0^\pi \left(\int_y^\pi \frac{\cos(x)}{x}dx\right) dy$
$= \int_0^\pi \left(\int_0^x \frac{\cos(x)}{x}dy\right) dx = \int_0^\pi \cos(x)dx = 0.$

(2) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function defined by $f(x,y,z) = 5z$. Find the integral $\int_\Omega f$, where Ω is the region bounded below by $z = x^2 - 3$, above by $z = x - 1$, and on the side by $y = 0$ and $y = 2$.

Solution:

$$\int_\Omega f = 5\int_0^2 \int_{-1}^2 \int_{x^2-3}^{x-1} zdzdxdy$$
$$= 5\int_{-1}^2 (x-1)^2 - (x^2-3)^2 dx = -63$$

(3) Find the volume of the solid in $\mathbb{R}^3$ bounded by $z = 2 - x^2 - y^2$ and $z = x^2 + y^2$.

Solution:

$$\text{Volume of the solid} = \int_{\{x^2+y^2\le 1\}} \int_{x^2+y^2}^{2-x^2-y^2} dzdxdy$$
$$= \int_{\{x^2+y^2\le 1\}} 2 - 2x^2 - 2y^2 dxdy$$
$$\int_{\{x^2+y^2\le 1\}} x^2 dxdy = \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2 dxdy$$
$$= \frac{2}{3}\int_{-1}^1 (1-x^2)^{3/2} dy = \frac{2}{3}\int_{-\pi/2}^{\pi/2} \cos^4\theta d\theta = \frac{\pi}{4}$$

Therefore,

$$\text{Volume of the solid} = \pi$$

(4) Let A be a rectangle in $\mathbb{R}^n$ and let $f, g : A \to \mathbb{R}$ be integrable functions. Show that $f + g$ is integrable and

$$\int_A (f+g) = \int_A f + \int_A g.$$

Solution: $L(f+g,P) = \sum_{S \in P} \inf_{x \in S}(f(x)+g(x))\mathbf{vol}(S) \geq \sum_{S \in P} \inf_{x \in S} f(x)\mathbf{vol}(S) + \sum_{S \in P} \inf_{x \in S} g(x)\mathbf{vol}(S) = L(f,P) + L(g,P)$ Similarly, $U(f+g,P') \leq U(f,P') + U(g,P')$

Therefore, let P_1 and P_2 be two partitions and let P be their common refinement. Then

$$\begin{aligned}&\sup_P L(f+g,P) \geq L(f+g,P)\\ &\geq L(f,P) + L(g,P) \geq L(f,P_1) + L(f,P_2).\end{aligned}$$

Take the sup in P_1 and P_2. Then

$$\sup_P L(f+g,P) \geq \int_A f + \int_A g$$

Similarly,

$$\inf_P U(f+g,P) \leq \int_A f + \int_A g$$

Finally, since $\inf_P U(f+g,P) \geq \sup_P L(f+g,P)$, we must have equality. So $f+g$ is integrable and $\int_A (f+g) = \int_A f + \int_A g$.
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