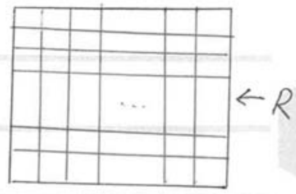
MATH2020A 19/9/2016.

Recall. $f: R \to \mathbb{R}$, $R \subseteq \mathbb{R}^n$.

$R = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$




$P_i = \{ a_1 = x_{i,0} < x_{i,1} < \dots < x_{i,n_i} = b_1 \}$.

Subrectangle of $P$ (not $R$) : $[x_{1,i_1-1}, x_{1,i_1}] \times \dots \times [x_{n,i_n-1}, x_{n,i_n}]$.

$P = (P_1, P_2, \dots, P_n)$ is a partition of $R$.

Upper sum : $\sum_{A: \text{subrec of } P} \sup_{x \in A} f(x) \operatorname{vol}(A) =: U(f, P)$.

Lower sum : $\sum_{A: \text{subrec of } P} \inf_{x \in A} f(x) \operatorname{vol}(A) =: L(f, P)$.

Recall. Let $B \subseteq \mathbb{R}$

$K \in \mathbb{R}$ is <u>the</u> least upper bound of $B$ if

(i) it is an upper bound (i.e. $x \leq K \ \forall x \in B$).

(ii) $\forall \varepsilon > 0$, $\exists x \in B$ s.t. $x > K - \varepsilon$.

We write $K = \sup_{x \in B} x$ [or in some books, $\sup B$].

Definition $f$ is integrable if $\sup_P L(f, P) = \inf_P U(f, P)$

i.e. $\exists$ sequence of $P_i$ s.t. $U(f, P_i) - L(f, P_i) \to 0$ as $i \to \infty$.

i.e. $\forall \varepsilon > 0$, $\exists$ partition $P$ s.t. $U(f, P) - L(f, P) < \varepsilon$.

Theorem (Fubini's Theorem)

$f: R \times R' \to \mathbb{R}$ is integrable, $R \subseteq \mathbb{R}^m$, $R' \subseteq \mathbb{R}^n$

Let $f_x(y) = f(x, y)$ [Fix $x$].

Then $\mathcal{U}(x) := \inf_P U(f_x, P)$, $\mathcal{L}(x) := \sup_P L(f_x, P)$.

are integrable and $\int_{R \times R'} f = \int_R \mathcal{U} = \int_R \mathcal{L}$

Proof

Let $P = (P_R, P_{R'})$. partition of $R \times R'$.

Let $A$ be a subrectangle of $P$.

Let $A = A_R \times A_{R'}$

$$U(f, P) = \sum_{A_R \times A_{R'} \text{ subrec of } P} \sup_{(x,y) \in A_R \times A_{R'}} f(x,y) \operatorname{vol}(A_R \times A_{R'}).$$

Exercise

$$= \sum_{\substack{A_R \text{ subrec} \\ \text{of } P_R}} \sup_{x \in A_R} \sum_{\substack{A_{R'} \text{ subrec} \\ \text{of } P_{R'}}} \sup_{y \in A_{R'}} f_x(y) \operatorname{vol}(A_{R'}) \operatorname{vol}(A_R).$$

$$\geq \sum_{A_R \text{ subrec of } P_R} \sup_{x \in A_R} \mathcal{U}(x) \operatorname{vol}(A_R) \qquad \text{[ By definition of } \mathcal{U}]$$

$$= U(\mathcal{U}, P_R)$$

Similarly, $L(f, P) \leq L(\mathcal{L}, P_R)$

Then

$$U(f, \tilde{P}) \geq U(\mathcal{U}, \tilde{P}_R)$$
$$\geq L(\mathcal{U}, P_R)$$
$$\geq L(\mathcal{L}, P_R)$$

$\because U(f_x, P) \geq L(f_x, \tilde{P}) \quad \forall P, \tilde{P}$ partition of $R'$

$\therefore \mathcal{U}(x) = \inf_P U(f_x, P) \geq \sup_{\tilde{P}} L(f_x, \tilde{P}) = \mathcal{L}(x)$

Since the inequality relation holds for all $\tilde{P}$ and $P_R$,

$$\inf_{\tilde{P}} U(f, \tilde{P}) \geq \inf_{\tilde{P}_R} U(\mathcal{U}, \tilde{P}_R) \geq \sup_{P_R} L(\mathcal{U}, P_R) \geq \sup_P L(f, P).$$

$\because f$ is integrable, $\therefore \inf_{\tilde{P}} U(f, \tilde{P}) = \sup_P L(f, P)$.

By Squeeze, $\inf_{\tilde{P}_R} U(\mathcal{U}, \tilde{P}_R) = \sup_{P_R} L(\mathcal{U}, P_R)$.

So $\mathcal{U}$ is integrable, and $\int_{R \times R'} f = \int_R \mathcal{U}$.

Similarly, $\mathcal{L}$ is integrable. So $\int_{R \times R'} f = \int_R \mathcal{U} = \int_R \mathcal{L}$ $\square$

Recall Let $f: R \to \mathbb{R}$, $C \subseteq R$.

$$\int_C f = \int_R f \chi_C, \quad \chi_C(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C. \end{cases}$$

[Note that $\chi$ is almost never continuous, so Fubini's Theorem should be "integrable" instead of "continuous"].

Definition $U$ is a set of measure $0$ if $\forall \varepsilon > 0$, $\exists$ countably many $C_i$ (rectangle) such that $U \subseteq \overset{\text{union}}{\bigcup} C_i$ and $\sum \operatorname{vol}(C_i) < \varepsilon$.

Theorem (*) $f: R \to \mathbb{R}$ bounded (i.e. $\exists M > 0$ s.t. $|f(x)| \leq M \ \forall x \in R$).

Then $f$ is integrable $\Leftrightarrow \{x \mid f \text{ is not cts at } x\}$ is of measure $0$.

Proof Skipped.

**Corollary** Suppose $f$ and $g : R \to \mathbb{R}$ are integrable. Then so is $f \cdot g$.

**Proof** If $f$ is cts at $x$ and $g$ is cts at $x$, then so is $f \cdot g$.

So $\{x \mid f\cdot g \text{ is not cts at } x\} \subseteq \{x \mid f \text{ is not cts at } x\} \cup \{x \mid g \text{ is not cts at } x\}$.

By the Theorem $(*)$, $f \cdot g$ is integrable. □

**Note** Union of countably infinite sets of measure zero is also measure zero.

**Corollary** $\chi_C$ is integrable $\Leftrightarrow \partial C$ is a set of measure zero.

**Proof** Claim: $\{x \mid \chi_C \text{ is not cts at } x\} = \partial C$.

Then with Theorem $(*)$, the proof is done.

Proof for Claim: Note that:

$x$ is an interior (exterior) point $\Rightarrow B(x, r)$ is in $C$ $(R \setminus C)$ for some $r > 0$.

$\Rightarrow \chi_C = 1$ $(0)$ in $B(x,r)$

$\Rightarrow \chi_C$ is continuous at $x$.

$\therefore \{x \mid \chi_C \text{ is not cts at } x\} \subseteq \partial C$.

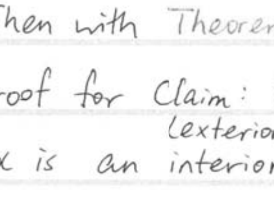
Now pick $x \in \partial C$, then $\exists\, x_N \in B(x, \frac{1}{N})$ s.t. $x_N \in C$.

$\exists\, y_N \in B(x, \frac{1}{N})$ s.t. $y_N \notin C$.


Now $\lim_{N\to\infty} \chi_C(y_N) = 0$, $\lim_{N\to\infty} \chi_C(x_N) = 1$. $\therefore \chi_C$ is not cts at $x$.

**Theorem** Definition of $\int_C f$ is independent on the choice of $R$.

**Proof**




Showing $\int_R f\chi_C = \int_{R''} f\chi_C$ is enough since then $\int_{R'} f\chi_C = \int_{R''} f\chi_C = \int_R f\chi_C$.

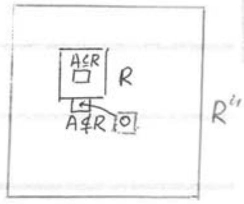
($R''$ is arbitrary).

Pick any partition $P'$ of $R''$, and construct $P$ partition of $R''$ such that $R$ is union of subrecs of $P$ (Adding green lines).

$$L(f\chi_c, P) \le L(f\chi_c, P') = \sum_{A:\text{subrec of } P} \inf_{x\in A} f(x)\chi_c(x)\,\mathrm{vol}(A).$$

$$= \sum_{A\subseteq R} \inf_{x\in A} f(x)\chi_c(x)\,\mathrm{vol}(A)$$

$$+ \sum_{A\not\subseteq R} \inf_{x\in A} f(x)\chi_c(x)\,\mathrm{vol}(A)$$



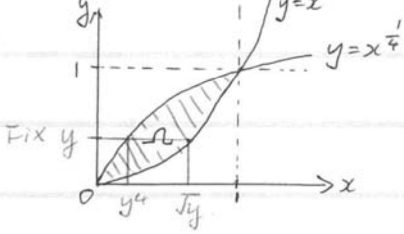

$$\le \int_R f\chi_c + \boxed{0} \leftarrow \inf \le f(x)\ \forall x \text{ and } \exists x \text{ s.t. } \chi_c(x) = 0, \text{ so } \inf \le 0$$

Since $P$ is arbitrary, $\int_{R''} f\chi_c \le \int_R f\chi_c$.

The other side of inequality holds by looking at upper sum.

Example 1. $\int_\Omega (x^{\frac{1}{2}} - y^2)\,dx\,dy$, $\Omega$ = region bounded by $y = x^{\frac{1}{4}}$ and $y = x^2$.

Solution

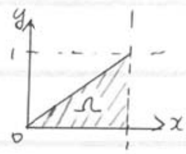



$$\int_\Omega (x^{\frac{1}{2}} - y^2)\,dx\,dy = \int_{[0,1]\times[0,1]} (x^{\frac{1}{2}} - y^2)\chi_\Omega\,dx\,dy.$$

$$\overset{F}{=} \int_0^1 \int_{y^4}^{\sqrt{y}} x^{\frac{1}{2}} - y^2\,dx\,dy.$$

$$= \dots = \frac{1}{7}.$$

Example 2 $\int_\Omega \cos(\frac{1}{2}\pi x^2)\,dx\,dy$




Solution Right : Fix $x$.

$$\text{Integral} = \int_0^1 \int_0^x \cos(\tfrac{1}{2}\pi x^2)\,dy\,dx$$

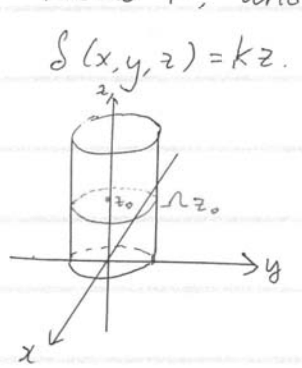
$$= \int_0^1 \boxed{x}\cos(\tfrac{1}{2}\pi x^2)\,dx \quad [x \text{ is crucial}]$$

$$= \frac{1}{\pi}.$$

Wrong : Fix $y$. We cannot (easily) compute it.

Example 3 Find the mass of a solid right circular cylinder of radius $r$, and height $h$ given that the mass density $\delta(x,y,z) = kz$.

Solution.




Fix $z$. Mass $= \int_\Omega \delta\,dV$.

$$\therefore \text{Mass} = \int_\Omega kz$$

$$= \int_0^h kz\left(\int_{\Omega_z} dx\,dy\right)dz$$

$$= \pi r^2 \int_0^h kz\,dz$$

$$= \frac{1}{2}kh^2\pi r^2 //$$

MATH 2020A 22/9/2016.

**Definition.** A subset $C \subseteq \mathbb{R}$ is said to be bounded if

$$\exists M \in \mathbb{R} \text{ s.t. } \forall x \in C,\ |x| \le M.$$

**Definition** A number $L$ is called <u>the</u> supremum (infimum) of $C$ if

(i) $L$ is an upper bound (a lower bound) (i.e. $\forall x \in C$, $x \le L$ $(\ge)$)

(ii) if $L'$ is also an upper bound (a lower bound), $L' \ge L$ $(\le)$.

$\Leftrightarrow$ (ii)' $\forall \varepsilon > 0$, $L - \varepsilon$ is not an upper bound (a lower bound) of $C$.

We write $L = \sup\limits_{x \in C} x$ $\left(\inf\limits_{x\in C}\right)$ (in some textbooks, $L = \sup C$ $(\inf)$)

**Terminology** Let $f: U \to \mathbb{R}$.

$$\sup_{x \in U} f(x) = \sup \{f(x) \mid x \in U\} = \sup f(U). \quad (\inf)$$

**Axiom** Any non-empty bounded subset of $\mathbb{R}$ has a sup and an inf.

**Proposition** $f(x) \le N$ $(\ge)$ $\forall x \in U \Leftrightarrow \sup\limits_{x \in U} f(x) \le N$ $\left(\inf, \ge\right)$

**Example** Prove that $\sup\limits_{x \in U} f(x) + \sup\limits_{y \in U} g(y) \ge \sup\limits_{z \in U} (f(z) + g(z))$.

**Solution** $\forall z \in U$, $f(z) \le \sup\limits_{x \in U} f(x)$.

$\forall z \in U$, $g(z) \le \sup\limits_{y \in U} g(y)$.

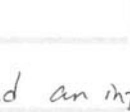
$\therefore \forall z \in U$, $f(z) + g(z) \le \sup\limits_{x \in U} f(x) + \sup\limits_{y \in U} g(y)$.

Then by Proposition, $\sup\limits_{z \in U} (f(z) + g(z)) \le \sup\limits_{x \in U} f(x) + \sup\limits_{y \in U} g(y)$.

**Review**

$f: R \to \mathbb{R}$ bounded, $R$: rectangle in $\mathbb{R}^n$, $P$: a partition of $R$.

i.e. $R = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$, $P = (P_1, P_2, \dots, P_n)$,




$P_i$ : a partition of $[a_i, b_i]$.

$$U(f, P) := \sum_{A \text{ subrec}} \left(\sup_{x \in A} f(x)\right) \mathrm{vol}(A), \quad L(f, P) := \sum_{A \text{ subrec}} \left(\inf_{x \in A} f(x)\right) \mathrm{vol}(A)$$

$f$ is said to be integrable on $R$ if $\sup\limits_P L(f, P) = \inf\limits_P U(f, P) =: \int_R f$.

Fubini's Theorem.

$f: R\times R' \to \mathbb{R}$ integrable. Let $f_x(y) = f(x,y)$ fix $x$.

Then $\mathcal{U}(x) := \inf_{P_{R'}} U(f_x, P_{R'})$ and $\mathcal{L}(x) := \sup_{P_{R'}} (f_x, P_{R'})$ are integrable and

$\int_R \mathcal{U} = \int_{R'} \mathcal{L} = \int_{R\times R'} f$

If $y \mapsto f_x(y)$ is integrable for each $x$, then $\int_R\int_{R'} f_x(y)\,dy\,dx = \int_{R\times R'} f$.

Theorem $f: R\to\mathbb{R}$ bounded. $f$ is integrable $\Leftrightarrow$ $\{x \mid f$ is not cts at $x\}$ is a set of measure 0.

Example Let $C=(0,1)$. $\because \chi_C$ is not cts at $\{a,b\}$, which is measure 0,

so $\chi_C$ is integrable.

(Recall $C\subseteq\mathbb{R}^n$ has $n$-dimensional measure 0 if $\forall\varepsilon>0$, $\exists$ a (countable) sequence of rectangles $c_1, c_2, \dots$ such that $C \subseteq \bigcup_{i=1}^{\infty} C_i$, and $\sum_{i=1}^{\infty} \mathrm{vol}(C_i) < \varepsilon$)

Corollary Suppose $f$ and $g: R\to\mathbb{R}$ are integrable. Then $f\cdot g$ is integrable.

Corollary Let $C\subseteq R$ (rectangle). $\chi_C: R\to\mathbb{R}$ is integrable $\Leftrightarrow$ $\partial C$ is measure 0.

Topology Let $C\subseteq\mathbb{R}^n$. $x$ is an interior point of $C$ if $\exists$ ball $B_r(x)\subseteq C$

$x$ is an exterior point of $C$ if $\exists$ ball $B_r(x) \subseteq \mathbb{R}^n\setminus C$.

$x\in\partial C$ if $x$ is not interior nor exterior point.

$C$ is open if $\mathrm{int}\, C = C$

$C$ is closed if $\partial C \subseteq C$.

$C$ is compact if $C$ is closed and bounded.

e.g. $C = [0,1]\cap\mathbb{Q}$, then $\partial C = [0,1]$. $\therefore \chi_C$ not integrable.

Integration on sets. Let $C\subseteq R$, $f: R\to\mathbb{R}$, then $\int_C f = \int_R f\cdot\chi_C$.

independent of the choice of $R$, shown in the last lecture.

this makes sense if $f$ is integrable and $\partial C$ has measure 0.

Change of variable (analogue to Substitution)

Proposition Let $C\subseteq R$ such that $C$ is closed and measure 0.

If $f: R\to\mathbb{R}$ is bounded and integrable, then $\int_C f = 0$.

Corollary If $\partial\Omega$ is measure zero, then $\int_\Omega f = \int_{\mathrm{int}\,\Omega} f$.

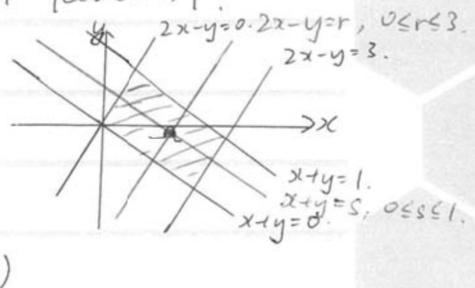
$\Omega$ is contained in R

Theorem (Change of variable formula).

Let $\varphi : \overset{\text{open}}{U} \to \overset{\text{open}}{V}$ in $\mathbb{R}^n$ be a diffeomorphism.

(i.e. $\varphi$ is continuously differentiable, invertible and $\varphi^{-1}$ is continuously diff.)

Let $f: U \to \mathbb{R}$ continuous, then $\int_{\varphi(U)} f = \int_U f \circ \varphi \, |\det(D\varphi)|$




Example. $f(x,y) = (x+y)^2$. Find $\int_\Omega f$.

Solution Define $\varphi : (0,3) \times (0,1) \to \Omega$ s.t.

$$\varphi^{-1}(x,y) = (2x-y,\ x+y). = (r,s).$$

$$\varphi(r,s) = \left(\frac{r+s}{3},\ s - \frac{r+s}{3}\right) = \left(\frac{r+s}{3},\ \frac{2s-r}{3}\right).$$

By change of variables,

$$\int_\Omega f = \int_{\varphi((0,3)\times(0,1))} f \overset{C}{=} \int_{(0,3)\times(0,1)} f \circ \varphi \, |\det(D\varphi)|$$

$$\overset{F}{=} \int_0^3 \int_0^1 \left(\frac{r+s}{3} + \left(\frac{-r+2s}{3}\right)\right)^2 |\det(D\varphi)| \, ds\, dr.$$

$$D\varphi(r,s) = \begin{array}{cc} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} & \begin{array}{l} \leftarrow \text{first entry in range} \\ \leftarrow \text{second} \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ \text{first} & \text{second entry in domain.} \end{array} & \end{array}$$

$\det D\varphi(r,s) = \frac{1}{3}$.

$\therefore$ Integral $= \frac{1}{3} \int_0^3 \int_0^1 (s)^2 \, ds\, dr$

$= \frac{1}{3}$