

12/9/2016.

In  $\mathbb{R}^2$ , rectangle  $R = [a_1, b_1] \times [a_2, b_2]$ Partition  $P = (P_1, P_2)$  where  $P_i$  is partition of  $[a_i, b_i]$ .

$$P_1 = \{a_1 = x_0 < x_1 < \dots < x_n = b_1\}, P_2 = \{a_2 = y_0 < y_1 < \dots < y_m = b_2\}$$

Example  $R = [1, 4] \times [1, 3]$ 

$$P_1 = \{1, 2, 3, 4\}, P_2 = \{1, \frac{3}{2}, 3\}.$$

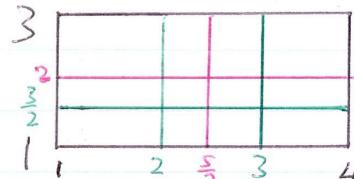
[ Partition = dividing the rectangle, so naturally = ]

Definition  $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$  is called a **subrectangle** of  $P$  (not  $R$ ).

Upper sum and lower sum.

Let  $f: R \xrightarrow{\text{rectangle}} \mathbb{R}$ . Let  $M_{i,j} = \sup_{x \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]} f(x)$  and  $m_{i,j} = \inf_{x \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]} f(x)$ Then upper sum  $U(f, P) = \sum_{\text{subrectangle of } P} M_{i,j} \cdot \text{vol}([x_{i-1}, x_i] \times [y_{j-1}, y_j])$ . (vol means area in 2D).lower sum  $L(f, P) = \sum_{\text{subrectangle of } P} m_{i,j} \cdot \text{vol}([x_{i-1}, x_i] \times [y_{j-1}, y_j])$ .Example Using  $R$  and  $P = (P_1, P_2)$  in the previous example. Let  $f(x,y) = x+y-2$ .

$$\begin{aligned} L(f, P) &= 0(2-1)(\frac{3}{2}-1) + 1(3-2)(\frac{3}{2}-1) + 2(4-3)(\frac{3}{2}-1) \\ &\quad + \frac{1}{2}(2-1)(3-\frac{3}{2}) + \frac{3}{2}(3-2)(3-\frac{3}{2}) + \frac{5}{2}(4-3)(3-\frac{3}{2}) = \frac{33}{4} \end{aligned}$$

Exercise  $U(f, P) = \frac{87}{4}$ .Definition  $P'$  is said to be a **refinement** of  $P$  if each subrectangle of  $P'$  is contained (is a subset) of  $P$ .e.g. If  $P = (P_1, P_2)$ ,  $P_1 = \{1, 2, 3, 4\}$ ,  $P_2 = \{1, \frac{3}{2}, 3\}$ , $P' = (P'_1, P'_2)$ ,  $P'_1 = \{1, 2, \frac{5}{2}, 3, 4\}$ ,  $P'_2 = \{1, \frac{3}{2}, 2, 3\}$ . $\leftarrow$  refine = further divide the rectangle. $P = (P_1, P_2)$  is a refinement of  $P'$ .Proposition Suppose  $P'$  is a refinement of  $P$ ,then (i)  $U(f, P') \leq U(f, P)$ (ii)  $L(f, P') \geq L(f, P)$ (iii)  $U(f, P_0) \geq L(f, P_1) \quad \forall \text{partition } P_0, P_1 \text{ of } R$ .

Proof

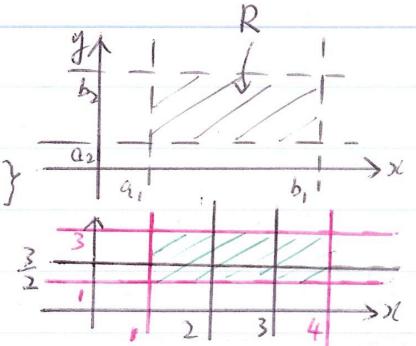
(i)  $\because$  If  $A' \subseteq A$ ,  $\sup_{x \in A'} f(x) \leq \sup_{x \in A} f(x)$ .  
A subrectangle of  $P$   
A subrectangle of  $P'$ 

$$\therefore U(f, P) = \sum_{\text{subrectangle of } P} \sup_{x \in A} f(x) \text{ vol}(A)$$

$$\geq \sum_{\substack{\text{subrectangle of } P \\ \text{contained in } A'}} \sup_{x \in A'} f(x) \text{ vol}(A'), \text{ where } \sum_{A'} \text{vol}(A') = \text{vol}(A)$$

$$\Rightarrow U(f, P) \geq U(f, P')$$

□



Proof (contd) (ii) Exercise (in fact consider  $f$  is OK).

(iii) Let  $P'$  be a refinement of  $P_0$  and  $P_1$ .

$$U(f, P_0) \geq U(f, P') \geq L(f, P') \geq L(f, P_1).$$

□

Integrable Definition

$f: R \rightarrow R$  is said to be **integrable** if

(i)  $f$  is bounded

$$(ii) \inf_P U(f, P) = \sup_P L(f, P).$$

$$\text{We write } \int_R f := \inf_P U(f, P) = \sup_P L(f, P).$$

Definition A set  $U$  is said to be of measure 0 if  $\forall \varepsilon > 0$ ,

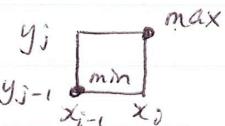
$\exists$  countably many <sup>cube</sup>  $C_i \subseteq R$  s.t.  $U \subseteq \bigcup_i C_i$  and  $\sum_i \text{vol}(C_i) < \varepsilon$ .

Fact  $f$  is continuous outside a set of measure 0 (means complement)  
 $\Leftrightarrow f$  is integrable

Example Let  $f(x, y) = x + y - 2$ ,  $R = [1, 4] \times [1, 3]$ ,

$$P_1 = \{1 = x_0 < \dots < x_n = 4\}, \quad P_2 = \{1 = y_0 < \dots < y_m = 3\} \rightarrow$$

Find  $\int_R f$ .



$$\begin{aligned} U(f, P) &= \sum_{j=1}^m \sum_{i=1}^n (x_i + y_j - 2)(x_i - x_{i-1})(y_j - y_{j-1}) \\ &= \sum_j \sum_i x_i (x_i - x_{i-1})(y_j - y_{j-1}) + \sum_j \sum_i y_j (x_i - x_{i-1})(y_j - y_{j-1}) \\ &\quad - 2 \sum_j \sum_i (x_i - x_{i-1})(y_j - y_{j-1}) \end{aligned}$$

[Note  $\sum_i (x_i - x_{i-1}) = 3$  and  $\sum_j (y_j - y_{j-1}) = 2$ ]

$$= 2 \sum_i (x_i)(x_i - x_{i-1}) + 3 \sum_j (y_j)(y_j - y_{j-1}) - 12.$$

Note that  $x_{i-1} \leq \frac{x_{i-1} + x_i}{2} \leq x_i$ ,  $y_{j-1} \leq \frac{y_{j-1} + y_j}{2} \leq y_j$ .

$$\therefore U(f, P) \geq 2 \sum_i \left( \frac{x_i + x_{i-1}}{2} \right) (x_i - x_{i-1}) + 3 \sum_j \left( \frac{y_j + y_{j-1}}{2} \right) (y_j - y_{j-1}) - 12 \geq L(f, P).$$

$$U(f, P) \geq \sum_i x_i^2 - x_{i-1}^2 + \frac{3}{2} \sum_j y_j^2 - y_{j-1}^2 - 12 \geq L(f, P).$$

Then  $L(f, P) \leq 15 \leq U(f, P)$ .

$\because f$  is continuous,  $\therefore f$  is integrable and  $\sup_P L(f, P) = \inf_P U(f, P)$ .

$$\int_R f = 15.$$

Characteristic Function  $\chi_U$ .

Definition Let  $U$  be a set.  $\chi_U = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$

Integration over non-rectangular set. For any bounded set  $U \subseteq R^n$ ,

$$\int_U f = \int_R f \chi_U.$$

Fact: Characteristic function is integrable if the boundary of  $U$  is of measure zero.

**Exercise** Let  $R$  and  $R'$  be two rectangles containing  $U$ .

If  $f \chi_u$  is integrable on  $R$ , then  $f \chi_u$  is integrable on  $R'$  and  $\int_R f \chi_u = \int_{R'} f \chi_u$ .

[i.e. integration is independent of rectangles chosen]

**Theorem (Fubini)** Let  $Q = R \times R'$ ,  $R \subseteq \mathbb{R}^n$ ,  $R' \subseteq \mathbb{R}^m$ ,  $R, R'$  are rectangles.

If  $f: Q \rightarrow \mathbb{R}$  is integrable,

let  $f(\underbrace{x_1, \dots, x_n}_{x}, \underbrace{y_1, \dots, y_m}_{y})$ ,  $y \mapsto f_x(y) = f(x, y)$ .

Then  $L(x) = \sup_P L(f_x, P)$  and  $U(x) = \inf_P U(f_x, P)$

are integrable over  $\mathbb{R}^n$ , " " $\mathbb{R}^m$ ,

and  $\int_R f = \int_R L = \int_{R \times R'} f$ .

**Corollary**

If  $f$  is continuous, then

$$\int_{R \times R'} f = \int_R \left( \int_{R'} f_x(y) dy \right) dx = \int_{R'} \left( \int_R f_y(x) dx \right) dy.$$

Assume that  $f, g$  are integrable

**Fact of integration** (i) Integration is linear, i.e.  $\int af + bg = a \int f + b \int g$ .

(ii) If  $f \leq g$ , then  $\int f \leq \int g$ .

(iii) Inclusion-exclusion principle  $\int_{S_1 \cup S_2} = \int_{S_1} + \int_{S_2} - \int_{S_1 \cap S_2}$ .

Assuming all characteristic functions involved are integrable

$$\text{Example 1: } \int_R x^4 - 2y \, dx \, dy$$

$$= \int_R (x^4 - 2y) \chi_R(x, y) \, dx \, dy$$

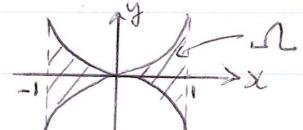
$$\stackrel{\text{Fubini}}{=} \int_{-1}^1 \left( \int_{-x^2}^{x^2} (x^4 - 2y) \chi_R(x, y) \, dy \right) dx$$

$$= \int_{-1}^1 \left( \int_{-x^2}^{x^2} (x^4 - 2y) \, dy \right) dx$$

$$= \int_{-1}^1 [x^4 y - y^2]_{-x^2}^{x^2} dx$$

$$= 2 \int_{-1}^1 x^6 dx$$

$$= \frac{4}{7}$$



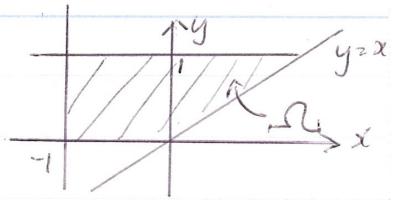
Make the domain become  $R$  by  $\chi_R$ .

Eliminate  $\chi_R$  as soon as possible after Fubini..

Treat  $x$  as constant  
(Similar to partial differentiation).

Example 2: Showing in Fubini,  $\iint f dy dx = \iint f dx dy$

$$\int_R (xy - y^3) dx dy$$



$$= \int_R (xy - y^3) \chi_R(x, y) dx dy.$$

[ $\chi_R \rightarrow R$  and  $\chi_R$  for Fubini.]

$$\stackrel{\text{Fubini}}{=} \int_{-1}^1 \left( \int_0^1 (xy - y^3) \chi_R(x, y) dy \right) dx$$

$$\text{Also } = \int_{-1}^0 \left( \int_0^1 (xy - y^3) dy \right) dx + \int_0^1 \left( \int_x^1 (xy - y^3) dy \right) dx$$

[separate the domain into two half for writing  $\int_a^b$ ]

$$= -\frac{23}{40}.$$

$$\rightarrow \stackrel{\text{Fubini}}{=} \int_0^1 \left( \int_{-1}^1 (xy - y^3) \chi_R(x, y) dx \right) dy$$

$$= \int_0^1 \left( \int_{-1}^y (xy - y^3) dx \right) dy \quad [\text{It is more convenient}]$$

$$= -\frac{23}{40}.$$

Remark: Try to find a direction such that the boundary can be expressed in an explicit formula, to prevent  $\chi_R$  being separated into 2 integrals or more.

## MATH2020A Advanced Calculus II.

15/9/2016

Recall

$f: R \xrightarrow{\text{rectangle}} R$  is said to be **integrable** if  $\inf_P U(f, P) = \sup_P L(f, P)$ .

We write  $\int f := \inf_P U(f, P) = \sup_P L(f, P)$ .

$f: A \xrightarrow{\text{closed}} R$ :  $\int f = \int_R f \chi_A$ , where  $\chi_A = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$ .

Theorem (Fubini's Theorem):

Let  $R \subseteq \mathbb{R}^n$ ,  $R' \subseteq \mathbb{R}^m$  be two rectangles. $f: Q = R \times R' \rightarrow \mathbb{R}$ 

$$(\vec{x}, \vec{y}) = (x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m)$$

Let  $f_x: R' \rightarrow \mathbb{R}$  s.t.  $f_x(y) := f(x, y)$  [Fixing  $x$ ].Then (i)  $U(x) := \inf_{P_{R'}} U(f_x, P_{R'})$ ,  $L(x) := \sup_{P_{R'}} L(f_x, P_{R'})$  are integrable.

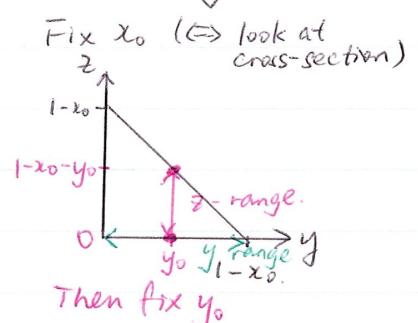
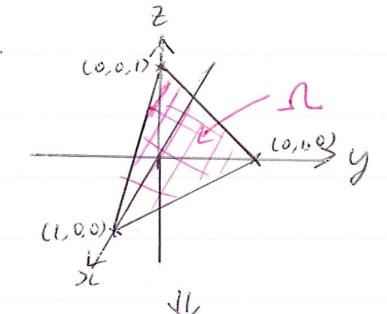
$$(ii) \int_R U = \int_R L = \int_Q f$$

(iii) If  $\forall x \in \mathbb{R}^n$ ,  $f_x$  is integrable (e.g.  $f$  is continuous),

$$\int_Q f = \int_R (\int_{R'} f_x(y) dy) dx.$$

Example Find the volume of the tetrahedron.

$$\begin{aligned} &= \int_R 1 = \int_{R=[0,1] \times [0,1] \times [0,1]} \chi_R \\ &\stackrel{(i)}{=} \int_{[0,1] \times [0,1]} \left( \int_0^1 \chi_R(x, y, z) dz \right) dx dy \\ &\stackrel{(ii)}{=} \int_0^1 \int_0^1 \int_0^1 \chi_R(x, y, z) dz dy dx. \\ &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz dy dx. \end{aligned}$$

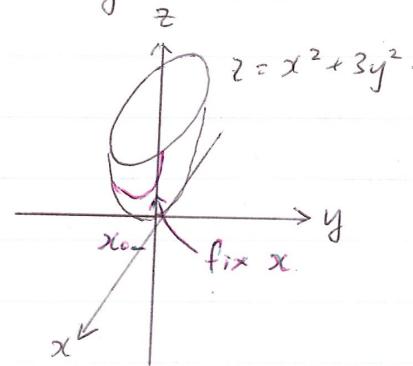
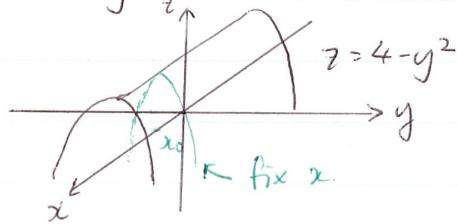


Note when solving (eliminating)  $\chi_R$ , remember to fix a good axis first (if we have  $y = x^2 + z$ ,  $x$  is the best to fix. In other words, an axis with linear terms in formula is advised to fix at last).

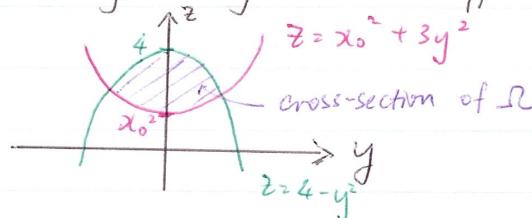
Draw the cross-section out and try to find the range of other variables.

2. Find the volume of the solid  $\Omega$  bounded by

$$z = 4 - y^2 \text{ and } z = x^2 + 3y^2.$$



when we fix  $x$  (with highest degree and disappear in one formula):



Range of  $x$ : Note the pink (V) line will shift according to  $x$ .

Cross-section of  $\Omega$  exists if  $x_0^2 \leq 4$ .

$$\therefore -2 \leq x \leq 2.$$

Then fix  $y$ . Range of  $y$  is given by those intersections  $\Leftrightarrow$  solving equation.

$$\begin{cases} z = x_0^2 + 3y^2 \\ z = 4 - y^2 \end{cases} \Rightarrow y = \pm \frac{1}{2}\sqrt{4 - x^2}$$

$$\therefore -\frac{1}{2}\sqrt{4 - x^2} \leq y \leq \frac{1}{2}\sqrt{4 - x^2}.$$

$z$  is from pink (V) to green (O), so  $x^2 + 3y^2 \leq z \leq 4 - y^2$

$$\text{Volume} = \int_{-2}^2 \int_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}} \int_{x^2+3y^2}^{4-y^2} dz dy dx$$

= ...

$$= \int_{-2}^2 \left[ 4y - x^2 y - \frac{4}{3}y^3 \right]_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}} dx$$

$$= \int_{-2}^2 (4\sqrt{4-x^2} - x^2\sqrt{4-x^2} - \frac{1}{3}(4-x^2)\sqrt{4-x^2}) dx \quad \leftarrow \text{don't try to expand the } (4-x^2). \text{ Keep it!}$$

$$= \frac{2}{3} \int_{-2}^2 (4-x^2)^{\frac{3}{2}} dx$$

Let  $x = 2\sin\theta$ ,  $dx = 2\cos\theta d\theta$

$$= \frac{32}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 4\theta d\theta$$

$$= \frac{32}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{\cos 2\theta + 1}{2} \right)^2 d\theta \quad \leftarrow \cos^2 \theta = \frac{\cos 2\theta + 1}{2}$$

= ...

$$= 4\pi,$$

## MATH2020A(Tutorial)

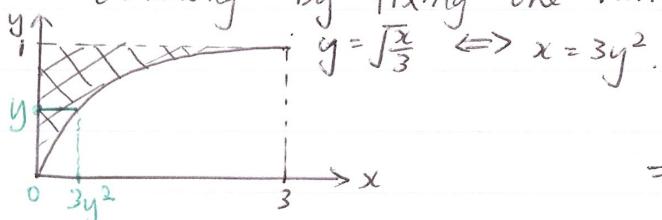
15/9/2016.

$$1. \int_0^3 \int_{\frac{x}{3}}^1 e^{y^3} dy dx.$$

**Thought** We might face some hard integral, then we could use Fubini's Theorem to help ourselves. Corollary of Fubini's Theorem:  
 $\iint dxdy = \int_R f = \iint dy dx$  for continuous  $f$ .

But we **MUST** be careful for the domain we are integrating (thinking by fixing one variable and drawing graphs).

Ans:



$$\begin{aligned} & \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 3, \sqrt{\frac{x}{3}} \leq y \leq 1\}, \\ & = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1, 0 \leq x \leq 3y^2\}. \end{aligned}$$

Fix y!

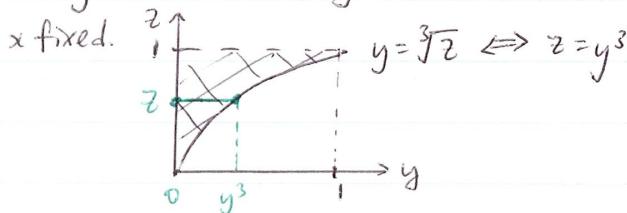
$$\begin{aligned} & \int_0^3 \int_{\frac{x}{3}}^1 e^{y^3} dy dx \\ & = \int_0^1 \int_0^{3y^2} e^{y^3} dx dy \\ & = \int_0^1 3y^3 e^{y^3} dy \\ & = \int_0^1 e^{y^3} dy \\ & = [e^{y^3}]_0^1 \\ & = e - 1. \end{aligned}$$

Fubini's Theorem.

 $e^{y^3}$  constant for  $x$ ,  $\int e^{y^3} dx = xe^{y^3} + C$ .

$$2. \int_0^1 \int_{\sqrt[3]{z}}^1 \int_0^{\ln 3} \frac{\pi e^{2x} \sin(\pi y^2)}{y^2} dx dy dz.$$

**Thought** This is a 3-D case. Note that we know  $\int \pi e^{2x} dx$  but do not know  $\int \frac{\sin(\pi y^2)}{y^2} dy$ . As a result, we can try to interchange  $dy$  and  $dz$  by Fubini's Theorem.



$$\begin{aligned} & \int_0^1 \int_{\sqrt[3]{z}}^1 \int_0^{\ln 3} \frac{\pi e^{2x} \sin(\pi y^2)}{y^2} dx dy dz \\ & = \int_0^1 \int_0^{y^3} \int_0^{\ln 3} \frac{\pi e^{2x} \sin(\pi y^2)}{y^2} dx dz dy \\ & = \frac{1}{2} \int_0^1 \int_0^{y^3} \left[ \frac{\pi \sin(\pi y^2)}{y^2} e^{2x} \right]_0^{\ln 3} dz dy \\ & = 4 \int_0^1 \int_0^{y^3} \frac{\pi \sin(\pi y^2)}{y^2} dz dy \\ & = 4 \int_0^1 y \pi \sin(\pi y^2) dy \\ & = 2 \int_0^1 \sin(\pi y^2) d(\pi y^2). \end{aligned}$$

Fubini's Theorem.

$$\int e^{2x} dx = \frac{1}{2} e^{2x} + C.$$

$$e^{2\ln 3} = 9, e^{2\ln 3} = 1, 9 - 1 = 8$$

 $\frac{\pi \sin(\pi y^2)}{y^2}$  constant for  $z$ .

(cont.).

$$\begin{aligned}
 2. &= [-2 \cos(\pi y^2)]_0^1 \\
 &= -2 (\cos(\pi) - \cos(0)) \\
 &= 4
 \end{aligned}$$

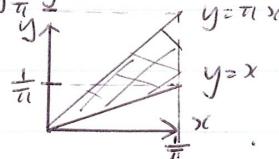
$$3. \int_0^{\frac{1}{\pi}} (\sin^{-1}(\pi x) - \sin^{-1}x) dx.$$

**Thought** It is often very troublesome to complete  $\int \sin^{-1}x dx$ . As a result, we try to use Fundamental Theorem of Calculus to convert this 1-D integral to 2-D one. Then using Fubini's Theorem to help us.  $dx$  first before  $dy$ .

$$\begin{aligned}
 &\int_0^{\frac{1}{\pi}} (\sin^{-1}\pi x - \sin^{-1}x) dx \\
 &= \int_0^{\frac{1}{\pi}} \int_{\pi x}^{\pi} (\sin^{-1}y)' dy dx \\
 &= \int_0^{\frac{1}{\pi}} \int_x^{\pi} \frac{1}{\sqrt{1-y^2}} dy dx \\
 &= \int_0^{\frac{1}{\pi}} \int_{\frac{y}{\pi}}^{\frac{1}{\pi}} \frac{1}{\sqrt{1-y^2}} dx dy + \int_{\frac{1}{\pi}}^1 \int_{\frac{y}{\pi}}^{\frac{1}{\pi}} \frac{1}{\sqrt{1-y^2}} dx dy \\
 &= 1 - \frac{1}{\pi} - \sqrt{1-\frac{1}{\pi^2}} + \frac{\sec^{-1}(\pi)}{\pi}
 \end{aligned}$$

Fundamental Theorem of Calculus:

$$\int_a^b (f(x))' dx = f(b) - f(a)$$

By ( $y = \sin^{-1}x$ ) implicit differentiation

During Computation:

$$\int \frac{y}{\sqrt{1-y^2}} = \sqrt{1-y^2}, \quad \int \frac{1}{\sqrt{1-y^2}} = \sin^{-1}y \quad (\text{From above}).$$

$$\frac{\pi}{2} - \sin^{-1}\left(\frac{1}{\pi}\right) = \sec^{-1}(\pi) \iff$$

