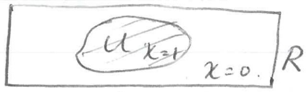
MATH 2020 Advanced Calculus II
8/9/2016

Overview of the course

We mainly deal with problems like $\int_U f$ (more computational),
where $f : \underset{\underset{\mathbb{R}^n,\ \text{e.g. } \mathbb{R}^2,\ \mathbb{R}^3}{\cap\!\!\!|}}{U} \to \mathbb{R}$

(1) Define $\int_{R=[a_1,b_1]\times[a_2,b_2]} f$, then for arbitrary (bounded) $U \subseteq \mathbb{R}^n$,



$f : \mathbb{R}^n \to \mathbb{R}$

$(U \subseteq R)$ $\int_U f = \int_R f \cdot \chi_U$, where $\chi_U(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$

(2) Fubini's Theorem

Under some condition of f:

$$\int_{[a_1,b_1]\times[a_2,b_2]} f(x,y)\,dx\,dy = \int_{a_2}^{b_2}\left(\int_{a_1}^{b_1} f(x,y)\,dx\right)dy = \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2} f(x,y)\,dy\right)dx$$

It means that we can treat one variable as constant and integrate another variable first. (Reduce 2D integration → two 1D)

Cor. Under condition of f, $\int_R f(x,y)\,dx\,dy = \int_R f(x,y)\,dy\,dx$.

(3) Change of variable

1D: $\int_{g(a)}^{g(b)} f(t)\,dt = \int_a^b f(g(s))\,g'(s)\,ds$.

Definition f is differentiable if $\exists\, Df(a)$ s.t. $\lim_{h\to 0} \frac{|f(x+h)-f(x)-Df(x)h|}{|h|} = 0$.

Let $\varphi : U \to V$, $U, V \subseteq \mathbb{R}^n$. If φ is differentiable and $\det D\varphi \neq 0$ (then $\exists\,\varphi^{-1}$, also differentiable),

$$\int_{g(U)} f = \int_U f\circ g \det D\varphi.$$

(4) Area of surface.   e.g. $(x_1, x_2) \mapsto (x_1, x_2, \sqrt{1-x_1-x_2})$

$D\varphi$ injective

upper hemisphere.

(5) Differential forms + integrate them over surfaces.

e.g. of 2-form. read as "wedge"

$$\alpha = a_{12}\, dx_1 \wedge dx_2 + a_{23}\, dx_2 \wedge dx_3 + a_{13}\, dx_1 \wedge dx_3.$$

$$a_{ij} : \mathbb{R}^3 \to \mathbb{R}.$$

$\int_M \alpha$. M is two dimensional
Integrate a 2-form over M

(6) Exterior derivative Some kind of differentiations for differential forms

Stokes theorem Exterior derivative

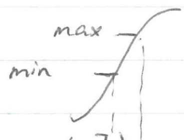
$$\int_M d\alpha = \int_{\partial M} \alpha.$$

Boundary of M

Integrable

$f : [a,b] \to \mathbb{R}$. Define $\int_a^b f(x)\,dx$?

Partition of $[a,b]$ $(\{a < t_1 < t_2 \ \cdots\ t_i < t_{i+1} \ \cdots < b\})$



↳ Sub-intervals $[t_{i-1}, t_i]$.

Let $m_i = \inf f([t_{i-1}, t_i])$, $M_i = \sup f([t_{i-1}, t_i])$.

Define lower sum $L(f,P) = \sum_{i=1}^{n} m_i (t_i - t_{i-1})$ [sum of area of rectangle

upper sum $U(f,P) = \sum_{i=1}^{n} M_i (t_i - t_{i-1})$

There are some functions which make L and U always unequa

e.g. Let $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$

$\forall$ partition P, $L(P,f) = 0$, $U(P,f) = 1$!

Then what is "integrable"? Note that if we cut finer, $U - L$ decreasin

If we refine the cutting: (P' is refined from P if $\forall I' \in P', \exists I \in P$ s.t. $I' \subseteq I$

$L(f, P') \geq L(f,P)$; $U(f,P') \leq U(f,P)$.

Definition $f : [a,b] \to \mathbb{R}$ is integrable if f is bounded and $\exists$ sequence of P_i (where P_{i+1} is refined from P_i $\forall i \in \mathbb{N}$) s.t.

$$\lim_{i\to\infty} (U(P_i, f) - L(P_i, f)) = 0.$$

We write $\int_a^b f(x)\,dx = \lim_{i\to\infty} U(P_i, f) = \lim_{i\to\infty} L(P_i, f)$.

Recall f is said to be continuous if $\lim_{x\to c} f(x) = f(c)$ $\forall c \in [a,b]$.

Exercise If f is continuous, then f is integrable. ($f : [a,b] \to \mathbb{R}$)