WEEK 9

(1) Let U be an open subset of $\mathbb{R}^n$. Let $f : U \to \mathbb{R}$ be a 0-form. The exterior derivative df of f is defined by
$$df = \frac{\partial f}{\partial x_1} dx_1 + ... + \frac{\partial f}{\partial x_n} dx_n.$$
(2) More generally, if $\alpha = \sum_{i_1 < ... < i_k} \alpha_{i_1,...,i_k} dx_{i_1} \wedge ... \wedge dx_{i_k}$, then
$$d\alpha = \sum_{i_1 < ... < i_k} \sum_j \frac{\partial \alpha_{i_1,...,i_k}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge ... \wedge dx_{i_k}.$$
(3) Note that $dx_i \wedge dx_i = 0$ and $dx_i \wedge dx_j = -dx_j \wedge dx_i$.
(4) e.g. if $\alpha = x_1 x_3 dx_2 \wedge dx_4$, then $d\alpha = x_3 dx_1 \wedge dx_2 \wedge dx_4 - x_1 dx_2 \wedge dx_3 \wedge dx_4$.
(5) e.g. If $\alpha = x_1 x_2 dx_2 \wedge dx_4 - x_2^2 dx_3 \wedge dx_4$, then $d\alpha = x_2 dx_1 \wedge dx_2 \wedge dx_4 - 2x_2 dx_2 \wedge dx_3 \wedge dx_4$.
(6) Proposition: $d^2\alpha = 0$.
(7) The generalized Stoke's theorem: $\int_M d\alpha = \int_{\partial M} \alpha$. We need to know what is ∂M (this is not the topological boundary of M) and we need to assign an orientation to ∂M so that $\int_{\partial M} \alpha$ makes perfect sense. We will do this in low dimensional cases only.
(8) Assume that M is an open set in $\mathbb{R}^2$. In this case ∂M is the topological boundary. Assume that the topological boundary is a finite union of closed smooth curves. (To be precise, the following condition is also needed: for each point x in ∂M there is an open ball B centred at x such that $\partial M \cap B$ is a graph of a smooth function and $B - \partial M$ consists of two connected components one of which is contained in M and the other one is outside the closure of M.)
(9) Under the above assumptions, we can define, for each x in ∂M, an outward pointing normal $n(x)$ such that $n(x)$ is perpendicular to the tangent space $T_x\partial M$ at x and $n(x)$ is pointing out of M. We orient ∂M by this outward pointing normal. This means that a vector v in the tangent space $T_x\partial M$ defines the boundary orientation of ∂M if $\{n(x), v\}$ coincides with the standard orientation $\{e_1, e_2\}$ on $\mathbb{R}^2$.
(10) e.g. If M is a disk D_r of radius r, then the boundary orientation on ∂M is counter clockwise orientation. If $M = D_R - D_r$, where

$r < R$, then ∂M consists of two components, one is ∂D_R, the circle of radius R, and the other one is ∂D_r, the circle of radius r. The boundary orientation induces by M on D_R is the counter clockwise rotation and that on D_r is the clockwise rotation.

(11) Green's theorem: $\alpha = f dx + g dy$ (is a 1-form defined on an open set containing the closure of M), $d\alpha = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$, and so
$$\int_{\partial M} f dx + g dy = \int_M \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy.$$

(12) Let $\alpha = (3x^2 + y)dx + (2x + y^3)dy$ and C is the circle of radius r centred at 0 equipped with the counter clockwise rotation. Find $\int_C \alpha$. Let D be the disk of radius r centred at 0. By Green's theorem $\int_C \alpha = \int_{\partial D} \alpha = \int_D d\alpha = \int_D dx \wedge dy = \pi r^2$.

(13) Let $\alpha = e^x \sin(y)dx + e^x \cos(y)dy$ and let C be the union of the semi-circle defined by $y = \sqrt{1 - x^2}$ and the line segment $\{(x, 0)|x \in [-1, 1]\}$. Find $\int_C \alpha$. One might start doing it by using the definition. But it can be done easily by Green's theorem since $d\alpha = 0$. It follows that $\int_C \alpha = \int_D d\alpha = 0$, where D is the upper half disk of radius 1.

(14) Let $\alpha = -\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$. Note that α is a 1-form on $\mathbb{R}^2 - \{0\}$, not on $\mathbb{R}^2$. If U is an open set such that 0 is not in U and $C = \partial U$ is a closed curve without self intersection. Note that $d\alpha = 0$. Then, by Green's theorem, $\int_C \alpha = \int_U d\alpha = 0$. On the other hand, if 0 is in U, then Green's theorem does not apply since α is not defined on U. Instead, let D_r be the disk of radius r. r is chosen such that D_r is contained in U. By Green's theorem,
$$0 = \int_{U - D_r} d\alpha = \int_{\partial(U - D_r)} \alpha = \int_{\partial U} \alpha - \int_{\partial D_r} \alpha.$$
Here ∂D_r is oriented by outward pointing normal of D_r which completely opposite to that induced by $U - D_r$. This accounts for the negative sign before $\int_{\partial D_r} \alpha$.

Let $\varphi : [0, 2\pi] \to \partial D_r$ be the map $\varphi(\theta) = (r \cos\theta, r \sin\theta)$. $D\varphi(\theta) = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix}$ and so
$$\int_{\partial U} \alpha = \int_{\partial D_r} \alpha = \int_0^{2\pi} 1 = 2\pi.$$