WEEK 8

(1) $dx_{i_1} \wedge ... \wedge dx_{i_n} : \mathbb{R}^N \times ... \times \mathbb{R}^N \to \mathbb{R}$ is a function defined by

$$dx_{i_1} \wedge ... \wedge dx_{i_n}(v^1, ..., v^n) = \det \begin{pmatrix} v^1_{i_1} & ... & v^k_{i_1} \\ \vdots & \ddots & \vdots \\ v^1_{i_n} & ... & v^k_{i_n} \end{pmatrix},$$

where $v^i = (v^i_1, ..., v^i_n)$.

(2) Let V be an open set in $\mathbb{R}^N$. A n-form on V is an object of the form

$$\alpha = \sum_{1 \leq i_1 < ... < i_n \leq N} \alpha_{i_1,...,i_n} dx_{i_1} \wedge ... \wedge dx_{i_n},$$

where $\alpha_{i_1,...,i_n} : V \to \mathbb{R}$ is a function.

For each fixed x in V, α defines a map $\alpha_x : \mathbb{R}^N \times ... \times \mathbb{R}^N \to \mathbb{R}$ by

$$\begin{aligned} &\alpha_x(v^1, ..., v^n) \\ &= \sum_{1 \leq i_1 < ... < i_n \leq N} \alpha_{i_1,...,i_n}(x) dx_{i_1} \wedge ... \wedge dx_{i_n}(v^1, ..., v^n). \end{aligned}$$

One should think of v^i as a tangent vector based at the point x and α takes n-tuples of such tangent vectors based at the same point and gives a number.

(3) Let U be an open set in $\mathbb{R}^n$ and let $\varphi : U \to \mathbb{R}^N$ be a parametrization of the set $M := \varphi(U)$ (i.e. φ is 1 - 1 and $D\varphi(x)$ has full rank for each x in U). Let α be a n-form. Define the integral of α over $\varphi(U)$ by

$$\int_{\varphi(U)} \alpha = \int_U \alpha_{\varphi(x)}(D\varphi(x)(e_1), ..., D\varphi(x)(e_n)).$$

The definition depends on the choice of parametrizations but only up to a sign.

(4) Let $\varphi_1 : U_1 \to \mathbb{R}^N$ and $\varphi_2 : U_2 \to \mathbb{R}^N$ be two parametrizations of M and assume that U_1 and U_2 are connected (i.e. any two points can be joined by a continuous path). The map $g := \varphi_2^{-1} \circ \varphi_1 : U_1 \to U_2$, which is a diffeomorphism, is called a transition map. By continuity, either $\det(Dg(x)) > 0$ or $\det(Dg(x)) < 0$ for all x. φ_1 and φ_2 are compatible if $\det(Dg(x)) > 0$. $\int_M \alpha$ is independent of the choice of compatible parametrizations.

(5) The proof involves the chain rule, the change of variable formula, and the following fact

$$\beta(Aw^1, ..., Aw^n) = \det(A)\beta(v^1, ..., v^n)$$

where A is a $n \times n$ matrix and β is a n-form.

This last fact, in turn, follows from the characterization of the determinant as the only multilinear and skew-symmetric function γ satisfying $\gamma(e_1, ..., e_n) = 1$.

(6) An orientation of a vector space W of dimension n is a choice of an ordered basis and two such choices $\{v^1, ..., v^n\}$ and $\{w^1, ..., w^n\}$ define the same orientation on W if the unique linear transformation L defined by

$$L(v_i) = w_i, \quad i = 1, ..., n$$

has positive determinant. Recall that the determinant of a linear transformation is defined as that of its matrix representation.

(7) Each parametrization $\varphi : U \to \mathbb{R}^N$ of M defines an orientation on M which is a smoothly varying orientation on each tangent space of M. This orienation is given by

$$\{D\varphi(x)(e_1), ..., D\varphi(x)(e_n)\}.$$

There are only two orientations of M if it exists. Two parametrizations define the same orientation if and only if they are compatible. So $\int_M \alpha$ is well-defined once an orientation of M is specified.

(8) e.g. $\varphi : (0,1) \to \mathbb{R}^2$, $\varphi(t) = (t, t^2)$, $\alpha = x_1x_2dx_1 + x_2^2dx_2$.

$$D\varphi(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$$

$$\int_M \alpha = \int_0^1 t \cdot t^2 \cdot 1 + (t^2)^2 \cdot 2t\, dt = 7/12.$$

(9) e.g. $\varphi : (0,1) \times (0,1) \to \mathbb{R}^3$, $\varphi(s,t) = (s+t, s^2, t^2)$, $\alpha = dx_1 \wedge dx_2 + x_2dx_1 \wedge dx_3$.

$$D\varphi(t) = \begin{pmatrix} 1 & 1 \\ 2s & 0 \\ 0 & 2t \end{pmatrix}$$

$$\int_M \alpha = \int_0^1 \int_0^1 \det\begin{pmatrix} 1 & 1 \\ 2s & 0 \end{pmatrix} + s^2 \det\begin{pmatrix} 1 & 1 \\ 0 & 2t \end{pmatrix} ds\, dt = -2/3.$$

(10) e.g. $U = \{(u,v,w) | (1-w)^2 > u^2+v^2 \text{ and } 0 < w < 1\}$, $\varphi : U \to \mathbb{R}^4$, $\varphi(u,v,w) = (u+v, u-v, w+v, w-v)$, $\alpha = x_2 x_4 dx_1 \wedge x_3 \wedge dx_4$.

$$D\varphi(t) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\int_M \alpha = \int_U (u-v)(w-v) \det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} du\, dv\, dw$$

$$= 2 \int_U (u-v)(w-v) du\, dv\, dw$$

$$= 2 \int_0^1 \int_0^{1-w} \int_0^{2\pi} r^2 (\cos\theta - \sin\theta)(w - r\sin\theta) d\theta\, dr\, dw = -\frac{\pi}{10}$$

(11) M is the triangle with vertices $(1,0), (0,1), (-1,0)$ oriented by the counterclockwise rotation. $\alpha = e^{x_2} dx_1 - \sin(\pi x_1) dx_2$. M is parametrized by $\varphi_i : (0,1) \to \mathbb{R}^2$, where

$$\varphi_1(t) = (1-t, t), D\varphi_1(t) = (-1, 1)^T$$
$$\varphi_2(t) = (-t, 1-t), D\varphi_2(t) = (-1, -1)^T$$
$$\varphi_3(t) = (-1+2t, 0), D\varphi_3(t) = (2, 0)^T.$$

$$\int_M \alpha = \int_{\varphi_1((0,1))} \alpha + \int_{\varphi_2((0,1))} \alpha + \int_{\varphi_3((0,1))} \alpha$$

$$= \int_0^1 e^t \cdot (-1) - \sin(\pi(1-t)) \cdot 1 dt$$

$$+ \int_0^1 e^{1-t} \cdot (-1) - \sin(\pi(-t)) \cdot (-1) dt$$

$$+ \int_0^1 e^0 \cdot 2 - \sin(\pi(-1+2t)) \cdot 0 dt = 4 - 2e - \frac{4}{\pi}.$$