WEEK 7

(1) Let A be a $n \times k$ matrix with $n \geq k$. Assume that A has full rank. The k-dimensional volume of $A([0,1]^k)$ is given by $\sqrt{\det(A^T A)}$.
(2) Note that $A^T A$ is invertible under the above assumptions.
(3) Note also that the ij-th entry of $A^T A$ is given by $\langle v_i, v_j \rangle$, where v_i is the i-th column of A.
(4) Let U be an open set in $\mathbb{R}^k$ and let $\varphi : U \to \mathbb{R}^n$ be a map which is 1 - 1 and $D\varphi(x)$ has full at each x in U. The k-dimensional volume of $\varphi(U)$ is defined by

$$\int_U \sqrt{\det(D\varphi(x)^T D\varphi(x))}.$$

(5) (U, φ) is called a parametrization of $M := \varphi(U)$. The k-dimensional volume of M is independent of the choice of parametrizations. The proof is done by a change of variables and the chain rule.
(6) Special case ($k = 1$): In this case,

$$\sqrt{\det(D\varphi(x)^T D\varphi(x))} = |\varphi'(x)|$$

and the 1-dimensional volume of $\varphi(U)$ is its arc-length.
(7) Special case ($k = 2$ and $n = 3$): In this case,

$$\sqrt{\det(D\varphi(x)^T D\varphi(x))} = |D\varphi(x)(e_1) \times D\varphi(x)(e_2)|,$$

where $\times$ is the cross product. The surface area, which is also the 2-dimensional volume, of $\varphi(U)$ is

$$\int_U |D\varphi(x)(e_1) \times D\varphi(x)(e_2)|.$$

(8) e.g. Area of sphere S of radius R: Let $\varphi : (0, 2\pi) \times (0, \pi) \to S$

$$\varphi(\theta, \alpha) = (R \sin\alpha \cos\theta, R \sin\alpha \sin\theta, R\cos\theta).$$

$$\det(D\varphi^T D\varphi) = R^4 \sin^2 \alpha$$

$$\text{Area of } S = \int_0^{2\pi} \int_0^{\pi} R^2 \sin\alpha d\alpha d\theta = 4\pi R^2$$

(9) Find the area of the surface M in $\mathbb{R}^4$ defined by $x_1^2 + x_2^2 = r_1^2$ and $x_3^2 + x_4^2 = r_2^2$.
Let $\varphi : (0, 2\pi)^2 \to M$ be the parametrization defined by

$$\varphi(\theta, \alpha) = (r_1 \cos\theta, r_1 \sin\theta, r_2 \cos\alpha, r_2 \sin\alpha).$$

$$\det(D\varphi^T D\varphi) = r_1^2 r_2^2$$
$$\text{Area of S} = 4\pi^2 r_1 r_2$$

(10) e.g. Let U be an open set in $\mathbb{R}^m$ and let $f : \mathbb{R}^m \to \mathbb{R}$ be a C^1 function. The m-dimensional volume of the graph $M = \{(x, f(x)) | x \in U\}$ is given by

$$\int_U \sqrt{1 + |Df(x)|^2} \tag{0.1}$$

Let $\varphi : U \to M$ be the parametrization defined by $(x, f(x))$. Then $D\varphi(x) = \begin{pmatrix} I \\ Df(x)^T \end{pmatrix}$. Then

$$\sqrt{\det(D\varphi(x)^T D\varphi(x))} = \sqrt{\det(I + Df(x)Df(x)^T)}$$

Note that the matrix $B := I + Df(x)Df(x)^T$ has eigenvector $Df(x)$ with eigenvalue $1 + |Df(x)|^2$.

$B - I$ has rank 1, so 1 is an eigenvalue of B with $m - 1$ dimensional eigenspace.

By spectral theorem $\det B$ is the product of eigenvalues of B, so $\det B = 1 + |Df(x)|^2$. Therefore, (0.1) holds.