WEEK 11

(1) The generalized Stokes' theorem: $\int_M d\alpha = \int_{\partial M} \alpha$. The Stokes' theorem is the special case when M is 2-dimensional surface with boundary in $\mathbb{R}^3$. We will assume that M is given by the image of a map $\varphi : \bar{U} \to \mathbb{R}^3$, where U is an open set in $\mathbb{R}^2$ such that the topological boundary ∂U is a finite union of closed curves and $\bar{U}$ is the union of U and ∂U (the closure of U). (More precisely, for each point x in ∂U, there is a neighbourhood B containing x such that $B \cap \partial U$ is a graph of a smooth function such that $B - \partial U$ contains two connected components, one in U and one outside. Moreover, there is an extension of φ to a smooth map defined on an open set containing $\bar{U}$ such that the extension is 1-1 and the its derivative has full rank everywhere.)

(2) In this case, $\alpha = f_1 dx + f_2 dy + f_3 dz$ and

$$d\alpha = \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx \wedge dy$$
$$- \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) dx \wedge dz + \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) dy \wedge dz.$$

Let $f = (f_1, f_2, f_3)$. The curl of f, denoted by $\operatorname{curl} f$ or $\nabla \times f$, is defined by

$$\operatorname{curl} f = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right).$$

By a discussion from last week, we have $\langle \operatorname{curl} f, n\rangle \, dS = d\alpha$, where n is the unit normal vector which defines the orientation of M.

(3) The orientation of the boundary ∂M is defined by the outward pointing normal N, which is a vector field defined along ∂M such that $N(x)$ is a unit vector contained in $T_x M$, is perpendicular to $T_x \partial M$, and is pointing out of M. The boundary ∂M is oriented by a vector field $\{T\}$ tangent to ∂M if and only if $\{N(x), T\}$ gives the orientation of $T_x M$.

(4) Stokes' theorem: $\int_M \langle \operatorname{curl} f, n\rangle \, dS = \int_M d\alpha = \int_{\partial M} \alpha$.

(5) One can define a 1-form ds on ∂M by the condition $ds(T) = 1$. It follows that $\alpha(T) = \langle f, T\rangle \, ds(T)$ and so $\alpha = \langle f, T\rangle \, ds$. So

Stokes' theorem is also given by

$$\int_M \langle \operatorname{curl} f, n\rangle \, dS = \int_M d\alpha = \int_{\partial M} \alpha = \int_{\partial M} \langle f, T\rangle \, ds.$$

(6) If the parametrization $\varphi : \bar{U} \to M$ induces the same orientation as the one given on M and v is the vector field tangent to ∂U which defines the boundary orientation of ∂U, then $d\varphi(v)$ defines the orientation of ∂M.

(7) e.g. Let M be the portion of the ellipsoid $2x^2 + 2y^2 + z^2 = 1$ that lies above the plane $z = \frac{1}{\sqrt{2}}$. Assume that M is oriented by the upward pointing normal. Find $\int_M \langle \operatorname{curl} f, n\rangle \, dS$, where $f = (-3y, 3x, z^4)$

Solution 1: Let $\varphi : U = \{(x, y) | x^2 + y^2 \le 1/4\} \to M$.

$$D\varphi(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-2x}{\sqrt{1-2x^2-2y^2}} & \frac{-2y}{\sqrt{1-2x^2-2y^2}} \end{pmatrix} =: (v_1, v_2).$$

$\alpha = -3ydx + 3xdy + z^4dz$, $d\alpha = 6dx \wedge dy$. So

$$\int_M \langle \operatorname{curl} f, n\rangle \, dS = \int_M d\alpha = 6 \int_U dx \wedge dy = \frac{3\pi}{2}$$

Solution 2: Let $\psi : (0, 2\pi) \to \partial M$ be the map defined by

$$\psi(\theta) = \left(\frac{1}{2}\cos\theta, \frac{1}{2}\sin\theta, \frac{1}{\sqrt{2}}\right).$$

One can use the definition and the picture to check that this gives the right orientation on ∂M. This also follows from (6). Therefore, by Stokes' theorem,

$$\int_M \langle \operatorname{curl} f, n\rangle \, dS = \int_{\partial M} \alpha = \frac{3}{4} \int_0^{2\pi} 1 = \frac{3\pi}{2}.$$

Solution 3: $\operatorname{curl} f = (0, 0, 6)$,

$$n = \frac{v_1 \times v_2}{|v_1 \times v_2|} = \frac{1}{|v_1 \times v_2|} \left(\frac{2x}{\sqrt{1 - 2x^2 - 2y^2}}, \frac{2y}{\sqrt{1 - 2x^2 - 2y^2}}, 1 \right),$$

$$|v_1 \times v_2|^2 = 1 + \frac{4x^2 + 4y^4}{1 - 2x^2 - 2y^2}.$$

Since $dS(v_1, v_2) = |v_1 \times v_2|$, it follows that

$$\int_M \langle \operatorname{curl} f, n\rangle \, dS = 6 \int_U 1 = \frac{3\pi}{2}.$$

(8) e.g. Let M be the upper half of the unit sphere $x^2+y^2+z^2=1$. Assume that M is oriented by the upward pointing normal. Find $\int_M \langle \operatorname{curl} f, n\rangle\, dS$, where $f=(z^2,-2x,y^3)$

Solution 1: Let $\varphi: U=\{(x,y)|x^2+y^2\le 1\}\to M$.

$$D\varphi(x,y)=\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-x}{\sqrt{1-x^2-y^2}} & \frac{-y}{\sqrt{1-x^2-y^2}} \end{pmatrix} =: (v_1,v_2).$$

$\alpha = z^2dx-2xdy+y^3dz$, $d\alpha=-2zdx\wedge dz-2dx\wedge dy+3y^2dy\wedge dz$. So

$$\int_M \langle \operatorname{curl} f, n\rangle\, dS = \int_M d\alpha$$
$$= \int_U 2y-2+\frac{3xy^2}{\sqrt{1-x^2-y^2}}dxdy.$$

By symmetry of U, $\int_U 2y+\frac{3xy^2}{\sqrt{1-x^2-y^2}}dxdy=0$. So

$$\int_M \langle \operatorname{curl} f, n\rangle\, dS = -2\int_U 1=-2\pi.$$

Solution 2: Let $\psi:(0,2\pi)\to\partial M$ be the map defined by

$$\psi(\theta)=(\cos\theta,\sin\theta,0).$$

By (6), the boundary orientation of ∂M coincides with the orientation defined by ψ. Therefore, by Stokes' theorem,

$$\int_M \langle \operatorname{curl} f, n\rangle\, dS=\int_{\partial M}\alpha=\int_0^{2\pi}-2\cos^2\theta d\theta=-2\pi.$$

Solution 3: $\operatorname{curl} f=(3y^2,2z,-2)$, $n=(x,y,z)$, $\langle \operatorname{curl} f, n\rangle = 3xy^2+2yz-2z$, and

$$dS(v_1,v_2)=\det\begin{pmatrix} x & 1 & 0 \\ y & 0 & 1 \\ \sqrt{1-x^2-y^2} & \frac{-x}{\sqrt{1-x^2-y^2}} & \frac{-y}{\sqrt{1-x^2-y^2}} \end{pmatrix}$$
$$=\frac{1}{\sqrt{1-x^2-y^2}}.$$

Again, by symmetry of U, it follows that

$$\int_M \langle \operatorname{curl} f, n\rangle\, dS=\int_U \frac{3xy^2+2y\sqrt{1-x^2-y^2}-2\sqrt{1-x^2-y^2}}{\sqrt{1-x^2-y^2}}$$
$$=-2\int_U 1=-2\pi.$$