WEEK 10

(1) The generalized Stokes' theorem: $\int_M d\alpha = \int_{\partial M} \alpha$. The divergence theorem is a special case when M is an open set in $\mathbb{R}^3$ such that ∂M is a union of finitely many smooth surface. (To be precise, the following condition is also needed: for each point x in ∂M there is an open ball B centred at x such that $\partial M \cap B$ is a graph of a smooth function and $B - \partial M$ consists of two connected components one of which is contained in M and the other one is outside the closure of M.)

(2) Under the above assumptions, we can define, for each x in ∂M, an outward pointing normal $n(x)$ such that $n(x)$ is perpendicular to the tangent space $T_x\partial M$ at x and $n(x)$ is pointing out of M. We orient ∂M by this outward pointing normal. This means that an ordered basis $\{v, w\}$ of the tangent space $T_x\partial M$ defines the boundary orientation of ∂M if $\{n(x), v, w\}$ coincides with the standard orientation $\{e_1, e_2, e_3\}$ on $\mathbb{R}^3$.

(3) In this case, $\alpha = F_1 dy \wedge dz - F_2 dx \wedge dz + F_3 dx \wedge dy$ and

$$\begin{aligned} d\alpha &= \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx \wedge dy \wedge dz \\ &=: \operatorname{div}(F) dx \wedge dy \wedge dz \end{aligned}$$

where $F = (F_1, F_2, F_3)$.

(4) The divergence theorem (version I):

$$\int_M \operatorname{div}(F) = \int_{\partial M} F_1 dy \wedge dz - F_2 dx \wedge dz + F_3 dx \wedge dy.$$

(5) Let $n(x)$ the unit normal vector field of a surface N which orients N. In other words, $n(x)$ has length one, $n(x)$ is perpendicular to T_xN for each x, and $\{v, w\}$ orients T_xN if and only if $\{n(x), v, w\}$ orients $\mathbb{R}^3$. Let dS be the 2-form on N defined by

$$dS(v_1, v_2) = \det \begin{pmatrix} n(x) & v_1 & v_2 \end{pmatrix},$$

where v_1 and v_2 are in T_xN.

(6) Note that dS acts on tangent vectors v in T_xN not a differential form defined on open subsets of $\mathbb{R}^3$. (Smoothness and exterior derivative of forms defined on surfaces are defined through charts, we will not discuss this in the course.)

(7) Fact: $\langle F(x), n(x)\rangle\, dS = F_1 dy \wedge dz - F_2 dx \wedge dz + F_3 dx \wedge dy$.

Proof: Decompose F by $F(x) = \langle F(x), n(x)\rangle\, n(x) + G(x)$, where $G(x)$ is in the tangent space $T_x N$.

$$
\begin{aligned}
&(F_1 dy \wedge dz - F_2 dx \wedge dz + F_3 dx \wedge dy)(v_1, v_2) \\
&= \det \begin{pmatrix} F(x) & v_1 & v_2 \end{pmatrix} \\
&= \langle F(x), n(x)\rangle \det \begin{pmatrix} n(x) & v_1 & v_2 \end{pmatrix} \\
&= \langle F(x), n(x)\rangle\, dS(v_1, v_2).
\end{aligned}
$$

(8) The divergence theorem (version II):

$$\int_M \operatorname{div}(F) = \int_{\partial M} \langle F, n\rangle\, dS,$$

where dS is defined by the outward pointing normal.

(9) Remark that $|v_1 \times v_2|$ is the volume of the parallelogram spanned by v_1 and v_2. $|\det(n(x), v_1, v_2)|$ is the volume of the parallelepiped spanned by $n(x)$, v_1, and v_2. Therefore,

$$|dS(v_1, v_2)| = |\det(n(x), v_1, v_2)| = |v_1 \times v_2|.$$

(10) Let M be the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane. Find $\int_M \langle F, n\rangle\, dS$, where $F(x, y, z) = (x, y, z)$ and n is the upward pointing normal.

Solution 1: Let $\varphi(x, y) = (x, y, 1 - x^2 = y^2)$ be a parametrization of M, where (x, y) is contained in the disk D of radius 1.

$$D\varphi(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2x & -2y \end{pmatrix} =: \begin{pmatrix} v & w \end{pmatrix}$$

Since $\det \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2x & -2y \end{pmatrix} = 1 > 0$, the orientation induced by φ coincides with that of defined by the upward pointing normal. Therefore,

$$
\begin{aligned}
&\int_M \langle F, n\rangle\, dS = \int_{\varphi(D)} x dy \wedge dz - y dx \wedge dz + z dx \wedge dy \\
&= \int_D x(2x) - y(-2y) + (1 - x^2 - y^2) = \int_D (1 + x^2 + y^2) \\
&= \int_0^{2\pi} \int_0^1 r(1 + r^2) dr d\theta = \frac{3\pi}{2}.
\end{aligned}
$$

Solution 2: $(1,0,-2x)\times(0,1,-2y)=(2x,2y,1)$ is pointing upward. So,

$$n=\frac{1}{\sqrt{1+4x^2+4y^2}}(2x,2y,1).$$

$$\begin{aligned}
\int_M \langle F,n\rangle\, dS &= \int_{\varphi(D)} \langle F,n\rangle\, dS \\
&= \int_D \frac{2x^2+2y^2+1-x^2-y^2}{\sqrt{(1+4x^2)(1+4y^2)}}dS(v,w) \\
&= \int_D \frac{1+x^2+y^2}{(1+4x^2)(1+4y^2)} \det\begin{pmatrix} 2x & 1 & 0 \\ 2y & 0 & 1 \\ 1 & -2x & -2y \end{pmatrix} \\
&= \int_D \frac{1+x^2+y^2}{(1+4x^2)(1+4y^2)}(1+4x^2+4y^2) \\
&= \int_D (1+x^2+y^2) = \frac{3\pi}{2}
\end{aligned}$$

(11) Find $\int_N \langle F,n\rangle\, dS$, where $F(x,y,z)=(x,y,0)$, N is the sphere of radius R centred at 0, and n is the outward pointing unit normal of the ball B of radius R centred at 0.

Solution 1: By the divergence theorem,

$$\int_N \langle F,n\rangle\, dS = \int_B \operatorname{div}(F) = 2\int_B 1 = \frac{8}{3}\pi R^3.$$

Solution 2: Let $\varphi : [0,2\pi]\times[0,\pi]\to N$ be defined by

$$\varphi(\theta,\phi)=\begin{pmatrix} R\sin\phi\cos\theta \\ R\sin\phi\sin\theta \\ R\cos\phi \end{pmatrix}.$$

Then

$$D\varphi = \begin{pmatrix} -R\sin\phi\sin\theta & R\cos\phi\cos\theta \\ R\sin\phi\cos\theta & R\cos\phi\sin\theta \\ 0 & -R\sin\phi \end{pmatrix} =: \begin{pmatrix} v_1 & v_2 \end{pmatrix}.$$

The outward pointing normal is $n = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix}$ and

$$
\begin{aligned}
&\det\begin{pmatrix} n & v_1 & v_2 \end{pmatrix} \\
&= \begin{pmatrix} \sin\phi\cos\theta & -R\sin\phi\sin\theta & R\cos\phi\cos\theta \\ \sin\phi\sin\theta & R\sin\phi\cos\theta & R\cos\phi\sin\theta \\ \cos\phi & 0 & -R\sin\phi \end{pmatrix} \\
&= -R^2\cos^2\phi\sin\phi - R^2\sin^3\phi = -R^2\sin\phi < 0.
\end{aligned} \tag{0.1}
$$

So the orientation induced by φ is opposite to that defined by the outward pointing normal.

Therefore,

$$
\begin{aligned}
&\int_N \langle F, n\rangle\, dS = -\int_{\varphi([0,2\pi]\times[0,\pi])} xdy\wedge dz - ydx\wedge dz \\
&= -\int_{[0,2\pi]\times[0,\pi]} -R^3\sin^3\phi\cos^2\theta - R^3\sin^3\phi\sin^2\theta \\
&= 2\pi R^3\int_0^\pi \sin^3\phi d\phi = 2\pi R^3\int_{-1}^1 1 - a^2 da = \frac{8\pi R^3}{3}.
\end{aligned}
$$

Here $a = \cos\phi$.

Solution 3: By (0.1), the orientation induced by φ is opposite to that defined by the outward pointing normal. Therefore,

$$
\begin{aligned}
&\int_N \langle F, n\rangle\, dS = -\int_{\varphi([0,2\pi]\times[0,\pi])} \langle F, n\rangle\, dS \\
&= -\int_{[0,2\pi]\times[0,\pi]} R\sin^2\phi dS(v_1, v_2) \\
&= \int_{[0,2\pi]\times[0,\pi]} R^3\sin^3\phi = \frac{8\pi R^3}{3}.
\end{aligned}
$$