HOMEWORK 7

For questions 1 to 4, U denotes an open set in $\mathbb{R}^3$.

(1) Let $f : U \to \mathbb{R}$ be a 0-form on U. Find the vector field $F = (F_1, F_2, F_3) : U \to \mathbb{R}^3$ such that $df = F_1 dx_1 + F_2 dx_2 + F_3 dx_3$. This vector field is called the gradient of f and it is usual denoted by ∇f.

(2) Let $G = (G_1, G_2, G_3) : U \to \mathbb{R}^3$ be a vector field and let $\alpha = G_1 dx_1 + G_2 dx_2 + G_3 dx_3$. Find the vector field $H = (H_1, H_2, H_3) : U \to \mathbb{R}^3$ such that $d\alpha = H_1 dx_2 \wedge dx_3 - H_2 dx_1 \wedge dx_3 + H_3 dx_1 \wedge dx_2$. This vector field is called the curl of v and it is usually denoted by $\nabla \times G$ or curl(G).

(3) Let $K = (K_1, K_2, K_3) : U \to \mathbb{R}^3$ be a vector field and let $\beta = K_1 dx_2 \wedge dx_3 - K_2 dx_1 \wedge dx_3 + K_3 dx_1 \wedge dx_2$. Find the function $g : U \to \mathbb{R}$ such that $d\beta = g dx_1 \wedge dx_2 \wedge dx_3$. This function is called the divergence of K and it is usually denoted by $\nabla \cdot K$ or div(K).

(4) Recall that $d^2 = 0$, where d is the exterior derivative. By using this fact, prove that $\nabla \times (\nabla f) = 0$ and $\nabla \cdot (\nabla \times G) = 0$.

(5) Do the following problems from the textbook: Section 16.2: 23, 24.

(6) Let V be an open set in $\mathbb{R}^2$ and let $F = (F_1, F_2) : V \to \mathbb{R}^2$ be a vector field on V. Let $r : (a, b) \to \mathbb{R}^2$ be a parametrization of a curve C in V. The line integral $\int_C F \cdot dr$ of F along C is defined by $\int_C F \cdot dr = \int_C \alpha$, where α is the 1-form $\alpha = F_1 dx + F_2 dy$. This is also the work done of the force F along the curve C. If the curve C is closed (i.e. it starts and ends end the same point), then it is also the circulation of the field F along C. Do the following problems from the textbook: Section 16.2: 25, 26, 28, Section 16.4: 20.

(7) Using the same notations as in the previous question, let β be the 1-form defined by $\beta = -F_2 dx + F_1 dy$. This is called the flux form. If the curve C is a closed curve without self intersection and is the boundary of an open set U. Then the flux of the field F across C is given by $\int_C \beta$, where C is equipped with the boundary orientation of U. Do the following problems from the textbook: Section 16.2: 30, 34, 36, 38.

(8) Do the following problems from the textbook: Section 16.4: 22, 24, 26, 28.

(9) Final comments on notations: In the textbook, the notations $\int_C F \cdot T ds$ and $\int_C F \cdot n\, ds$ are also used for the line integral and flux, respectively.

The vector T is simply $\frac{1}{|r'(t)|} r'(t)$ (using the same notations as in question 6). ds is a "1-form on C" taking vectors tangent to C and giving a number. It is defined by $ds_x(v) = \det(n(x) \quad v)$, where $n(x)$ is the unit vector perpendicular to the tangent space $T_x C$ such that $\{n(x), T\}$ defines the standard orientation in $\mathbb{R}^2$ (i.e. the outward pointing unit normal in the case of question 7). It follows that $ds(T) = 1$ and $F \cdot T ds(T) = F \cdot T = \alpha(T)$ (α is defined in question 6). Therefore, $F \cdot T ds = \alpha$.

For the flux form β defined in question 7, $F \cdot n\, ds(T) = F \cdot n$, $\beta(T) = \det(F \quad T)$, and $F = (F \cdot n)\, n + (F \cdot T)\, T$. Therefore,

$$\beta(T) = \det((F \cdot n)\, n + (F \cdot T)\, T \quad T) = F \cdot n\, ds(T).$$

Hence $\beta = F \cdot n\, ds$.