HOMEWORK 6

For question 1 - 4, U is an open set in $\mathbb{R}^n$ and $\varphi : U \to \mathbb{R}^N$ is a parametrization of $\varphi(U)$. Find $\int_{\varphi(U)} \alpha$.

(1) $n = 1$, $N = 3$, $U = (-1, 1)$,
$\varphi(x) = (\sin(x), \cos(x), x)$,
$\alpha = xdy + ydz$.

(2) $n = 2$, $N = 3$, $U = (-1, 1) \times (-1, 1)$,
$\varphi(u, v) = (u^2, u + v, v^3)$,
$\alpha = x\, dy \wedge dz$.

(3) $n = 2$, $N = 3$,
$U = \{(u, v) | u > 0, v > 0, \text{ and } u + v < 2\}$,
$\varphi(u, v) = (uv, u^2 + v^2, u - v, \log(u + v + 1))$,
$\alpha = x_1\, dx_2 \wedge dx_3 + x_2\, dx_3 \wedge dx_4$.

(4) $n = 3$, $N = 4$,
$U = \{(u, v, w) | u > 0, v > 0, w > 0, \text{ and } u + v + w < 3\}$,
$\varphi(u, v, w) = (uv, u^2 + w^2, u - v, w)$,
$\alpha = x_2\, dx_1 \wedge dx_3 \wedge dx_4$.

(5) Let U be an open set in $\mathbb{R}^3$ and let $F : U \to \mathbb{R}^3$ be a vector field on U. The work form W of $F = (F_1, F_2, F_3)$ is defined by

$$W = F_1 dx_1 + F_2 dx_2 + F_3 dx_3.$$

Let I be an open interval and let $\varphi : I \to \mathbb{R}^3$ be a parametrization of a curve $M := \varphi(I)$. The work done of the force F along M is defined by

$$\int_{\varphi(I)} W.$$

What is the work done of

$$F(x, y, z) = (y, -x, 0)$$

along $\varphi(I)$, where $I = (0, 4\pi)$ and $\varphi(t) = (\cos(t), \sin(t), t)$?

(6) Let U be an open set in $\mathbb{R}^3$ and let $F : U \to \mathbb{R}^3$ be a vector field on U. The flux form Φ of $F = (F_1, F_2, F_3)$ is defined by

$$\Phi = F_1 dx_2 \wedge dx_3 - F_2 dx_1 \wedge dx_3 + F_3 dx_1 \wedge dx_2.$$

Let V be an open set in $\mathbb{R}^2$ and let $\varphi : V \to \mathbb{R}^3$ be a parametrization of a surface $M := \varphi(V)$. The flux of the force F across the surface M is defined by

$$\int_{\varphi(V)} \Phi.$$

What is the flux of

$$F(x, y, z) = (x, y^2, z)$$

across $\varphi(V)$, where $V = (0,1)\times(0,1)$ and $\varphi(u,v) = (u^2, uv, v^2)$?

(7) Let U be an open set in $\mathbb{R}^3$ and let $f : U \to \mathbb{R}$ be a function. The mass density form ρ of f is defined by

$$\rho = f dx_1 \wedge dx_2 \wedge dx_3.$$

The mass of the solid U with mass density f is defined by

$$\int_U \rho.$$

Here U is oriented by $\{e_1, e_2, e_3\}$.

Let $V = (0, r) \times (0, 2\pi) \times (0, 2\pi)$ and let $\varphi : V \to \mathbb{R}^3$ be defined by

$$\varphi(u, v, w) = ((R + u\cos(v))\cos(w), (R + u\cos(v))\sin(w), u\sin(v))$$

where $r < R$.

What is the mass of $U = \varphi(V)$ if the mass density of U is given by $f(x, y, z) = x^2 + y^2$?