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14. $f(x,y,z)=y^2+4z-16=0 \Rightarrow \nabla f = 2y\mathbf{j}+4\mathbf{k} \Rightarrow |\nabla f| = 2\sqrt{y^2+4}$ and $p=k \Rightarrow |\nabla f\cdot p|=4$

$\Rightarrow d\sigma = \frac{\sqrt{y^2+4}}{4}\,dx\,dy \Rightarrow \iint_S G\,d\sigma = \int_{-4}^{4}\int_0^1 (x\sqrt{y^2+4})\left(\frac{\sqrt{y^2+4}}{2}\right)dx\,dy = \int_{-4}^{4}\frac{1}{4}(y^2+4)\,dy = \frac{56}{3}$

16. $f(x,y,z)=x^2+y-z=0 \Rightarrow \nabla f = 2x\mathbf{i}+\mathbf{j}-\mathbf{k} \Rightarrow |\nabla f| = \sqrt{2}\sqrt{2x^2+1}$ and $p=k$

$\Rightarrow |\nabla f\cdot p| = 1 \Rightarrow d\sigma = \sqrt{2}\sqrt{2x^2+1}\,dx\,dy \Rightarrow \iint_S G\,d\sigma = \int_0^1\int_0^1 x\sqrt{2}\sqrt{2x^2+1}\,dx\,dy = \frac{3\sqrt{6}-\sqrt{2}}{3}$

18. $f(x,y,z)=x+y-1 \Rightarrow \nabla f = \mathbf{i}+\mathbf{j} \Rightarrow |\nabla f| = \sqrt{2}$ and $p=j \Rightarrow |\nabla f\cdot p| = 1 \Rightarrow d\sigma = \sqrt{2}\,dz\,dx$

$\Rightarrow \iint_S G\,d\sigma = \int_0^1\int_0^1 (x-(1-x)-z)\sqrt{2}\,dz\,dx = \sqrt{2}\int_0^1\int_0^1 (2x-z-1)\,dz\,dx = \sqrt{2}\int_0^1 (2x-\frac{3}{2})\,dx = -\frac{\sqrt{2}}{2}$

38. $g(x,y,z)=x^2+y^2-z=0 \Rightarrow \nabla g = 2x\mathbf{i}+2y\mathbf{j}-\mathbf{k} \Rightarrow |\nabla g| = \sqrt{4(x^2+y^2)+1} \Rightarrow n = \frac{2x\mathbf{i}+2y\mathbf{j}-\mathbf{k}}{\sqrt{4(x^2+y^2)+1}}$

$p=k \Rightarrow |\nabla g\cdot p| = 1 \Rightarrow d\sigma = \sqrt{4(x^2+y^2)+1}\,dx\,dy$ $\quad\Rightarrow F\cdot n = \frac{8x^2+8y^2-2}{\sqrt{4(x^2+y^2)+1}}$

$\Rightarrow \text{Flux} = \iint_R \frac{8x^2+8y^2-2}{\sqrt{4(x^2+y^2)+1}}\cdot\sqrt{4(x^2+y^2)+1}\,dx\,dy = \iint_R (8(x^2+y^2)-2)\,dx\,dy = \int_0^{2\pi}\int_0^1 (8r^2-2)\,r\,dr\,d\theta = 2\pi$

40. $g(x,y,z)=y-\ln x=0 \Rightarrow \nabla g = -\frac{1}{x}\mathbf{i}+\mathbf{j} \Rightarrow |\nabla g| = \sqrt{\frac{1}{x^2}+1} = \frac{\sqrt{1+x^2}}{x}$ since $1\le x\le e$

$\Rightarrow n = \frac{-\mathbf{i}+x\mathbf{j}}{\sqrt{1+x^2}} \Rightarrow F\cdot n = \frac{2xy}{\sqrt{1+x^2}}$. $p=j \Rightarrow |\nabla g\cdot p| = 1 \Rightarrow d\sigma = \frac{\sqrt{1+x^2}}{x}\,dx\,dz$

$\Rightarrow \text{Flux} = \iint_R \frac{2xy}{\sqrt{1+x^2}}\cdot\frac{\sqrt{1+x^2}}{x}\,dx\,dz = \int_0^1\int_1^e 2y\,dx\,dz = \int_1^e\int_0^1 2\ln x\,dz\,dx = \int_1^e 2\ln x\,dx = 2$

42. $g(x,y,z)=x^2+y^2+z^2-25=0 \Rightarrow \nabla g = 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k} \Rightarrow |\nabla g| = 10 \Rightarrow n = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{5}$

$\Rightarrow F\cdot n = \frac{x^2z}{5}+\frac{y^2z}{5}+\frac{z}{5}$. $p=k \Rightarrow |\nabla g\cdot p| = 2z$ since $z\ge 0 \Rightarrow d\sigma = \frac{10}{2z}\,dx\,dy$

$\Rightarrow \text{Flux} = \iint_{cap} F\cdot n\,d\sigma = \iint_R \left(\frac{x^2z}{5}+\frac{y^2z}{5}+\frac{z}{5}\right)\cdot\frac{5}{z}\,dx\,dy = \iint_R (x^2+y^2+1)\,dx\,dy = \int_0^{2\pi}\int_0^4 (r^2+1)\,r\,dr\,d\theta$

$= \int_0^{2\pi} 72\,d\theta = 144\pi$

§16.8

25.(a) $\text{div}(gF) = \frac{\partial}{\partial x}(gM)+\frac{\partial}{\partial y}(gN)+\frac{\partial}{\partial z}(gP) = g\frac{\partial M}{\partial x}+M\frac{\partial g}{\partial x}+g\frac{\partial N}{\partial y}+N\frac{\partial g}{\partial y}+g\frac{\partial P}{\partial z}+P\frac{\partial g}{\partial z}$

$\nabla\cdot(gF) = g\,\text{div}F + F\cdot\nabla g = g(\nabla\cdot F)+F\cdot\nabla g$

(b) $\nabla\times(gF) = \left[\frac{\partial}{\partial y}(gP)-\frac{\partial}{\partial z}(gN)\right]\mathbf{i}+\left[\frac{\partial}{\partial z}(gM)-\frac{\partial}{\partial x}(gP)\right]\mathbf{j}+\left[\frac{\partial}{\partial x}(gN)-\frac{\partial}{\partial y}(gM)\right]\mathbf{k}$

$= \left[\frac{\partial g}{\partial y}P-\frac{\partial g}{\partial z}N+g\frac{\partial P}{\partial y}-g\frac{\partial N}{\partial z}\right]\mathbf{i}+\left[\frac{\partial g}{\partial z}M-\frac{\partial g}{\partial x}P+g\frac{\partial M}{\partial z}-g\frac{\partial P}{\partial x}\right]\mathbf{j}$

$+\left[\frac{\partial g}{\partial x}N-\frac{\partial g}{\partial y}M+g\frac{\partial N}{\partial x}-g\frac{\partial M}{\partial y}\right]\mathbf{k}$

$= \left[\frac{\partial g}{\partial y}P-\frac{\partial g}{\partial z}N\right]\mathbf{i}+\left[\frac{\partial g}{\partial z}M-\frac{\partial g}{\partial x}P\right]\mathbf{j}+\left[\frac{\partial g}{\partial x}N-\frac{\partial g}{\partial y}M\right]\mathbf{k}$

$+ g\left[\left(\frac{\partial P}{\partial y}-\frac{\partial N}{\partial z}\right)\mathbf{i}+\left(\frac{\partial M}{\partial z}-\frac{\partial P}{\partial x}\right)\mathbf{j}+\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\mathbf{k}\right]$

$= \nabla g\times F + g(\nabla\times F)$

26. (a) Let $\mathbf{F}_1 = M_1\mathbf{i} + N_1\mathbf{j} + P_1\mathbf{k}$ and $\mathbf{F}_2 = M_2\mathbf{i} + N_2\mathbf{j} + P_2\mathbf{k}$

$$\Rightarrow a\mathbf{F}_1 + b\mathbf{F}_2 = (aM_1 + bM_2)\mathbf{i} + (aN_1 + bN_2)\mathbf{j} + (aP_1 + bP_2)\mathbf{k}$$

$$\Rightarrow \nabla\cdot(a\mathbf{F}_1 + b\mathbf{F}_2) = \left(a\frac{\partial M_1}{\partial x} + b\frac{\partial M_2}{\partial x}\right) + \left(a\frac{\partial N_1}{\partial y} + b\frac{\partial N_2}{\partial y}\right) + \left(a\frac{\partial P_1}{\partial z} + b\frac{\partial P_2}{\partial z}\right)$$

$$= a\left(\frac{\partial M_1}{\partial x} + \frac{\partial N_1}{\partial y} + \frac{\partial P_1}{\partial z}\right) + b\left(\frac{\partial M_2}{\partial x} + \frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z}\right) = a(\nabla\cdot\mathbf{F}_1) + b(\nabla\cdot\mathbf{F}_2)$$

(b) Define $\mathbf{F}_1$ and $\mathbf{F}_2$ as in part a

$$\Rightarrow \nabla\times(a\mathbf{F}_1 + b\mathbf{F}_2) = \left[\left(a\frac{\partial P_1}{\partial y} + b\frac{\partial P_2}{\partial y}\right) - \left(a\frac{\partial N_1}{\partial z} + b\frac{\partial N_2}{\partial z}\right)\right]\mathbf{i} + \left[\left(a\frac{\partial M_1}{\partial z} + b\frac{\partial M_2}{\partial z}\right) - \left(a\frac{\partial P_1}{\partial x} + b\frac{\partial P_2}{\partial x}\right)\right]\mathbf{j}$$

$$+ \left[\left(a\frac{\partial N_1}{\partial x} + b\frac{\partial N_2}{\partial x}\right) - \left(a\frac{\partial M_1}{\partial y} + b\frac{\partial M_2}{\partial y}\right)\right]\mathbf{k}$$

$$= a\left[\left(\frac{\partial P_1}{\partial y} - \frac{\partial N_1}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M_1}{\partial z} - \frac{\partial P_1}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y}\right)\mathbf{k}\right] + b\left[\left(\frac{\partial P_2}{\partial y} - \frac{\partial N_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M_2}{\partial z} - \frac{\partial P_2}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N_2}{\partial x} - \frac{\partial M_2}{\partial y}\right)\mathbf{k}\right]$$

$$= a\nabla\times\mathbf{F}_1 + b\nabla\times\mathbf{F}_2$$

(c) $$\mathbf{F}_1\times\mathbf{F}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ M_1 & N_1 & P_1 \\ M_2 & N_2 & P_2 \end{vmatrix} = (N_1P_2 - P_1N_2)\mathbf{i} - (M_1P_2 - P_1M_2)\mathbf{j} + (M_1N_2 - N_1M_2)\mathbf{k}$$

$$\Rightarrow \nabla\cdot(\mathbf{F}_1\times\mathbf{F}_2) = \nabla\cdot\left[(N_1P_2 - P_1N_2)\mathbf{i} - (M_1P_2 - P_1M_2)\mathbf{j} + (M_1N_2 - N_1M_2)\mathbf{k}\right]$$

$$= \tfrac{\partial}{\partial x}(N_1P_2 - P_1N_2) - \tfrac{\partial}{\partial y}(M_1P_2 - P_1M_2) + \tfrac{\partial}{\partial z}(M_1N_2 - N_1M_2)$$

$$= \left(P_2\frac{\partial N_1}{\partial x} + N_1\frac{\partial P_2}{\partial x} - N_2\frac{\partial P_1}{\partial x} - P_1\frac{\partial N_2}{\partial x}\right) - \left(M_1\frac{\partial P_2}{\partial y} + P_2\frac{\partial M_1}{\partial y} - P_1\frac{\partial M_2}{\partial y} - M_2\frac{\partial P_1}{\partial y}\right)$$

$$+ \left(M_1\frac{\partial N_2}{\partial z} + N_2\frac{\partial M_1}{\partial z} - N_1\frac{\partial M_2}{\partial z} - M_2\frac{\partial N_1}{\partial z}\right)$$

$$= M_2\left(\frac{\partial P_1}{\partial y} - \frac{\partial N_1}{\partial z}\right) + N_2\left(\frac{\partial M_1}{\partial z} - \frac{\partial P_1}{\partial x}\right) + P_2\left(\frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y}\right) + M_1\left(\frac{\partial N_2}{\partial z} - \frac{\partial P_2}{\partial y}\right) + N_1\left(\frac{\partial P_2}{\partial x} - \frac{\partial M_2}{\partial z}\right) + P_1\left(\frac{\partial M_2}{\partial y} - \frac{\partial N_2}{\partial x}\right)$$

$$= \mathbf{F}_2\cdot\nabla\times\mathbf{F}_1 - \mathbf{F}_1\cdot\nabla\times\mathbf{F}_2$$

27. Let $\mathbf{F}_1 = M_1\mathbf{i} + N_1\mathbf{j} + P_1\mathbf{k}$ and $\mathbf{F}_2 = M_2\mathbf{i} + N_2\mathbf{j} + P_2\mathbf{k}$.

(a) $$\mathbf{F}_1\times\mathbf{F}_2 = (N_1P_2 - P_1N_2)\mathbf{i} + (P_1M_2 - M_1P_2)\mathbf{j} + (M_1N_2 - N_1M_2)\mathbf{k}$$

$$\Rightarrow \nabla\times(\mathbf{F}_1\times\mathbf{F}_2) = \left[\tfrac{\partial}{\partial y}(M_1N_2 - N_1M_2) - \tfrac{\partial}{\partial z}(P_1M_2 - M_1P_2)\right]\mathbf{i} + \left[\tfrac{\partial}{\partial z}(N_1P_2 - P_1N_2) - \tfrac{\partial}{\partial x}(M_1N_2 - N_1M_2)\right]\mathbf{j}$$

$$+ \left[\tfrac{\partial}{\partial x}(P_1M_2 - M_1P_2) - \tfrac{\partial}{\partial y}(N_1P_2 - P_1N_2)\right]\mathbf{k}$$

consider the **i**-component only: $\tfrac{\partial}{\partial y}(M_1N_2 - N_1M_2) - \tfrac{\partial}{\partial z}(P_1M_2 - M_1P_2)$

$$= N_2\frac{\partial M_1}{\partial y} + M_1\frac{\partial N_2}{\partial y} - M_2\frac{\partial N_1}{\partial y} - N_1\frac{\partial M_2}{\partial y} - M_2\frac{\partial P_1}{\partial z} - P_1\frac{\partial M_2}{\partial z} + P_2\frac{\partial M_1}{\partial z} + M_1\frac{\partial P_2}{\partial z}$$

$$= \left(N_2\frac{\partial M_1}{\partial y} + P_2\frac{\partial M_1}{\partial z}\right) - \left(N_1\frac{\partial M_2}{\partial y} + P_1\frac{\partial M_2}{\partial z}\right) + \left(\frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z}\right)M_1 - \left(\frac{\partial N_1}{\partial y} + \frac{\partial P_1}{\partial z}\right)M_2$$

$$=\left(M_2\frac{\partial M_1}{\partial x}+N_2\frac{\partial M_1}{\partial y}+P_2\frac{\partial M_1}{\partial z}\right)-\left(M_1\frac{\partial M_2}{\partial x}+N_1\frac{\partial M_2}{\partial y}+P_1\frac{\partial M_2}{\partial z}\right)+\left(\frac{\partial M_2}{\partial x}+\frac{\partial N_2}{\partial y}+\frac{\partial P_2}{\partial z}\right)M_1-\left(\frac{\partial M_1}{\partial x}+\frac{\partial N_1}{\partial y}+\frac{\partial P_1}{\partial z}\right)M_2$$

Now, **i**-comp of $(\mathbf{F}_2\cdot\nabla)\mathbf{F}_1=\left(M_2\frac{\partial}{\partial x}+N_2\frac{\partial}{\partial y}+P_2\frac{\partial}{\partial z}\right)M_1=\left(M_2\frac{\partial M_1}{\partial x}+N_2\frac{\partial M_1}{\partial y}+P_2\frac{\partial M_1}{\partial z}\right)$;

likewise, **i**-comp of $(\mathbf{F}_1\cdot\nabla)\mathbf{F}_2=\left(M_1\frac{\partial M_2}{\partial x}+N_1\frac{\partial M_2}{\partial y}+P_1\frac{\partial M_2}{\partial z}\right)$;

i comp of $(\nabla\cdot\mathbf{F}_2)\mathbf{F}_1=\left(\frac{\partial M_2}{\partial x}+\frac{\partial N_2}{\partial y}+\frac{\partial P_2}{\partial z}\right)M_1$ and **i**-comp of $(\nabla\cdot\mathbf{F}_1)\mathbf{F}_2=\left(\frac{\partial M_1}{\partial x}+\frac{\partial N_1}{\partial y}+\frac{\partial P_1}{\partial z}\right)M_2$.

Similar results hold for the **j** and **k** components of $\nabla\times(\mathbf{F}_1\times\mathbf{F}_2)$. In summary, since the corresponding components are equal, we have the result $\nabla\times(\mathbf{F}_1\times\mathbf{F}_2)=(\mathbf{F}_2\cdot\nabla)\mathbf{F}_1-(\mathbf{F}_1\cdot\nabla)\mathbf{F}_2+(\nabla\cdot\mathbf{F}_2)\mathbf{F}_1-(\nabla\cdot\mathbf{F}_1)\mathbf{F}_2$

(b) Here again we consider only the **i**-component of each expression. Thus, the **i**-comp of $\nabla(\mathbf{F}_1\cdot\mathbf{F}_2)$

$$=\frac{\partial}{\partial x}(M_1M_2+N_1N_2+P_1P_2)=\left(M_1\frac{\partial M_2}{\partial x}+M_2\frac{\partial M_1}{\partial x}+N_1\frac{\partial N_2}{\partial x}+N_2\frac{\partial N_1}{\partial x}+P_1\frac{\partial P_2}{\partial x}+P_2\frac{\partial P_1}{\partial x}\right)$$

i-comp of $(\mathbf{F}_1\cdot\nabla)\mathbf{F}_2=\left(M_1\frac{\partial M_2}{\partial x}+N_1\frac{\partial M_2}{\partial y}+P_1\frac{\partial M_2}{\partial z}\right)$,

i-comp of $(\mathbf{F}_2\cdot\nabla)\mathbf{F}_1=\left(M_2\frac{\partial M_1}{\partial x}+N_2\frac{\partial M_1}{\partial y}+P_2\frac{\partial M_1}{\partial z}\right)$,

i-comp of $\mathbf{F}_1\times(\nabla\times\mathbf{F}_2)=N_1\left(\frac{\partial N_2}{\partial x}-\frac{\partial M_2}{\partial y}\right)-P_1\left(\frac{\partial M_2}{\partial z}-\frac{\partial P_2}{\partial x}\right)$, and

i-comp of $\mathbf{F}_2\times(\nabla\times\mathbf{F}_1)=N_2\left(\frac{\partial N_1}{\partial x}-\frac{\partial M_1}{\partial y}\right)-P_2\left(\frac{\partial M_1}{\partial z}-\frac{\partial P_1}{\partial x}\right)$.

Since corresponding components are equal, we see that

$\nabla(\mathbf{F}_1\cdot\mathbf{F}_2)=(\mathbf{F}_1\cdot\nabla)\mathbf{F}_2+(\mathbf{F}_2\cdot\nabla)\mathbf{F}_1+\mathbf{F}_1\times(\nabla\times\mathbf{F}_2)+\mathbf{F}_2\times(\nabla\times\mathbf{F}_1)$, as claimed.

§16.8

6. $F = x^2 i + y^2 j + z^2 k \Rightarrow \nabla F = 2x + 2y + 2z$

(a) $\text{Flux} = \int_0^1 \int_0^1 \int_0^1 (2x+2y+2z)\,dx\,dy\,dz = 3$

(b) $\text{Flux} = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (2x+2y+2z)\,dx\,dy\,dz = 0$

(c) $\text{Flux} = \int_0^1 \int_0^{2\pi} \int_0^2 (2r\cos\theta + 2r\sin\theta + 2z)\,r\,dr\,d\theta\,dz = \int_0^1 \int_0^{2\pi} \left(\frac{16}{3}\cos\theta + \frac{16}{3}\sin\theta + 4z\right) d\theta\,dz = 4\pi$

14. $F = \dfrac{xi + yj + zk}{\sqrt{x^2+y^2+z^2}} \Rightarrow \nabla \cdot F = \dfrac{2}{\sqrt{x^2+y^2+z^2}}$

$\text{Flux} = \int_0^{2\pi} \int_0^{\pi} \int_1^2 \frac{2}{r}(r^2 \sin\phi)\,dr\,d\phi\,d\theta = 12\pi$

16. $F = \ln(x^2+y^2)\,i - \left(\frac{2z}{x}\tan^{-1}\frac{y}{x}\right) j + z\sqrt{x^2+y^2}\,k \Rightarrow \nabla \cdot F = \dfrac{2x}{x^2+y^2} - \dfrac{2z}{x^2+y^2} + \sqrt{x^2+y^2}$

$\text{Flux} = \int_0^{2\pi} \int_1^{\sqrt{2}} \int_{-1}^2 \left(\frac{2r\cos\theta}{r^2} - \frac{2z}{r^2} + r\right) dz\,r\,dr\,d\theta = 2\pi\left(-\frac{3}{2}\ln 2 + 2\sqrt{2} - 1\right)$

Quiz3 (Version 1).

(1). $\int_C 3y\,dx + 2x\,dy = \iint_D (-3+2)\,dx\,dy = -\int_0^{\pi} \int_0^{\sin x} dy\,dx = -2$

(2) $g(x,y,z) := \sqrt{x^2+y^2} - z = 0 \Rightarrow \nabla g = \frac{x}{\sqrt{x^2+y^2}} i + \frac{y}{\sqrt{x^2+y^2}} j - k \Rightarrow |\nabla g| = \sqrt{2}\ \cancel{\Rightarrow n =}$

$\Rightarrow n = \frac{x}{\sqrt{2}\sqrt{x^2+y^2}} i + \frac{y}{\sqrt{2}\sqrt{x^2+y^2}} j - \frac{1}{\sqrt{2}} k \Rightarrow F \cdot n = -\frac{\sqrt{x^2+y^2} + z^2}{\sqrt{2}}$

$p = k \Rightarrow |\nabla g \cdot p| = 1 \Rightarrow dS = \sqrt{2}\,dx\,dy$

$\Rightarrow \int_M \langle F, n\rangle\,dS = \int_0^{2\pi} \int_1^2 -\frac{r+r^2}{\sqrt{2}} \cdot \sqrt{2}\ \ r\,dr\,d\theta = -\frac{73\pi}{6}$

Version 2

(1) $\int_C (x-y)\,dx + (x+y)\,dy = \iint_D (1+1)\,dx\,dy = 2\int_0^1 \int_0^{1-x} dy\,dx = 1$

(2) $g(x,y,z) := \sqrt{x^2+y^2} - z \Rightarrow \nabla g = \frac{x}{\sqrt{x^2+y^2}} i + \frac{y}{\sqrt{x^2+y^2}} j - k \Rightarrow |\nabla g| = \sqrt{2}$

$\Rightarrow n = \frac{x}{\sqrt{2}\sqrt{x^2+y^2}} i + \frac{y}{\sqrt{2}\sqrt{x^2+y^2}} j - \frac{1}{\sqrt{2}} k \Rightarrow F \cdot n = \frac{x^2 y}{\sqrt{2}\sqrt{x^2+y^2}} + \frac{z}{\sqrt{2}}$

$p = k \Rightarrow |\nabla g \cdot p| = 1 \Rightarrow dS = \sqrt{2}\,dx\,dy$

$\Rightarrow \int_M \langle F, n\rangle\,dS = \int_0^{2\pi} \int_0^1 \left(\frac{r^2\cos^2\theta \cdot r\sin\theta}{\sqrt{2}\,r} + \frac{r}{\sqrt{2}}\right) \cdot \sqrt{2}\ r\,dr\,d\theta = \frac{2\pi}{3}$