Week 7

Topics covered:

L'Hôpital's Rule (as application of the Cauchy Mean Value Theorem)

Topics to cover:

- Applications of Lagrange's Mean Value Theorem
- (i) $f'(x) > 0, < 0$ *then increasing, decreasing*
  (ii) curve sketching, (the terminologies "critical point", "point of inflection")
- To prove all the "mean value theorems", we need the Extreme Value Theorem.
- Proof of $n = 0$ TT by way of LMVT
- Proof of $n = 1$ TT by way of CMVT & LMVT

First Application of LMVT

Let $f(x)$ be a function such that $f'(x) > 0\ \forall x \in Domain$, then $f(x)$ *is strictly increasing*. (i.e. whenever $s < t$ in the domain of $f(x)$, then $f(s) < f(t)$).

Remark: Similarly, we have $f'(x) < 0\ \forall x \in Domain$, then $f(x)$ *is strictly decreasing*.

Proof:

Want: Show $f(s) < f(t)$ *whenever* $s < t$.

Consider $\frac{f(s)-f(t)}{s-t}$.

By using the LMVT, we have $\exists d$ between s & t such that $\frac{f(s)-f(t)}{s-t} = f'(d)$.

Now there are two cases for the word "between", i.e. it means either $s < d < t$ or $t < d < s$

In either case, the denominator is "negative" because $s < t$ if and only if $s - t < 0$.

Hence it follows that the numerator is also "negative" to make the quotient $\frac{f(s)-f(t)}{s-t} > 0\ldots$

Hence $f(s) < f(t)$ as required.

Application of "$f'(x) > 0$ implies "strictly increasing" " in "curve sketching.

Recall our old example:

Example: Sketch $f(x) = \frac{1}{x(x-1)}$.

We have (i) $\lim_{x\to-\infty} f(x) = 0^+$, (ii) $\lim_{x\to 0^-} f(x) = +\infty$, (iii) $\lim_{x\to 0^+} f(x) = -\infty$, (iv) $\lim_{x\to 1^-} f(x) = -\infty$, (v) $\lim_{x\to 1^+} f(x) = +\infty$, (vi) $\lim_{x\to\infty} f(x) = 0^+$.

What we don't know

How many "bumps" are there?

By "bump", we mean "local maximum/minimum points". These points are found by

(i) Looking for point(s) c satisfying the equation $f'(c) = 0$. (Such point c is called a "critical point")

(ii) Checking whether the function is "strictly increasing/decreasing" when $x < c$ and near c; "strictly decreasing/increasing" when $x > c$ and near c. (The first case means "c is a (local) maximum point", the second case means "c is a (local) minimum point".)

(iii) Sometimes, one can also check for (local) max/min points by considering $f''(c) < 0$ *or* > 0.

Curve Sketching Example continued

Consider $f(x) = \frac{1}{x(x-1)}$,

Then $f'(x) = \frac{-(\frac{d}{dx}x(x-1))}{x^2(x-1)^2} = \frac{-[(x-1)+x]}{x^2(x-1)^2} = \frac{-(2x-1)}{x^2(x-1)^2}$

Solving $f'(x) = 0$ gives $x = \frac{1}{2}$.

Now when $x < \frac{1}{2}$ and near to it, we get $f'(x) > 0$. When $x > \frac{1}{2}$ and near to it, we get $f'(x) < 0$. So $x = \frac{1}{2}$ is a local maximum point.

Definition: A point c in the domain is called a point of "inflection" (or "inflexion"), if for $x < c$ ((and near the point c), $f''(x) > 0$ and for $x > c$ (and near the point c), $f''(x) < 0$. (or vice versa, i.e. $f''(x) < 0$ first, then $f''(x) > 0$ next).
(In short, it means "$f''(x)$ changes "sign" about the point c")

Question: Does this function have a point of inflexion?

Answer: No, because $\frac{d}{dx}\left(\frac{-(2x-1)}{x^2(x-1)^2}\right) = -\left(\frac{x^2(x-1)^2 2-(2x-1)\frac{d}{dx}[x^2(x-1)^2]}{x^4(x-1)^4}\right)$

$$-\left(\frac{x^2(x-1)^2 2-(2x-1)[2x(x-1)\{2x-1\}]}{x^4(x-1)^4}\right) = 0$$

$$x^2(x-1)^2 2 = (2x-1)^2[2x(x-1)]$$
$$x(x-1) = (2x-1)^2$$
$$x^2 - x = 4x^2 - 4x + 1$$
$$0 = 3x^2 - 3x + 1$$

I.e. $x = \frac{3\pm\sqrt{9-12}}{6}$

This has no solution. So there is no "inflection points".

Proof of Taylor's Theorem continued

- **Done via EVT &**
- **LMVT &**
- **CMVT**

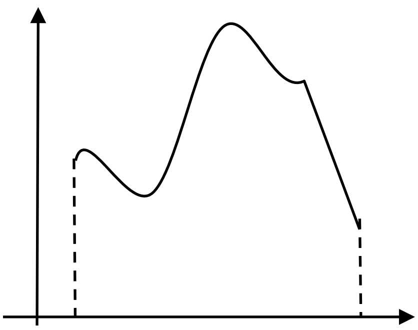
We will explain them one by one.

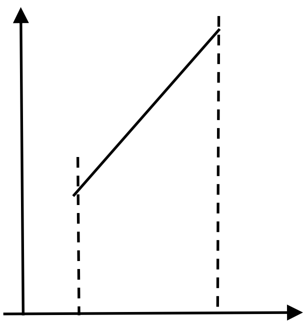
A technical result:

*** Extreme Value Theorem**

Assumption: $f: [a,b] \to R$ is a continuous function.

Conclusion: f has global (or "absolute") max/min values.





Left picture –a global max. attained inside the interval, global min. attained at the right end-point.

Right picture – both global max. and min. are attained at end-points.

Question: What do we mean by global abs. max value?

A point c in the domain of $f(x)$ is called a "global" maximum point, if $f(c) \geq f(x), \ \forall x$ in domain of $f(x)$

The value $f(c)$ is called the "global" max. value of the function.

Similarly, one can define "global minimum point" and "value".

Terminology:

One can summarized both maximum and minimum in the word "extremum".

(Using this word, we can say "The EVT guarantees that any continuous function defined on $[a,b]$ has global extrema ("extrema" is the plural of "extremum")")

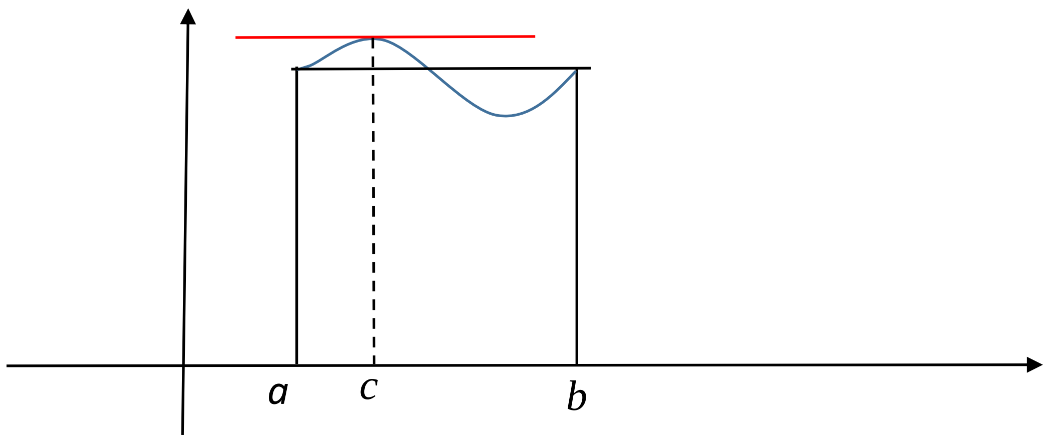
Remarks (for the EVT):

- Continuity is sufficient (we don't need "differentiability")
- Domain must be of the form $[a,b]$. ($(a,b],[a,b),(a,b)$, etc. wouldn't work!)
- Pure existence theorem (it doesn't tell you how to find the max/min points)
- Using this, we can prove LMVT (via the RT)

Question: Do you remember the statement of RT?

Rolle's Theorem

The following picture explains Rolle's Theorem:



Rolles' Theorem says: "If a function $f(x)$ satisfies (1), (2), (3) below, then $\exists c \in (a,b)$ such that $f'(c) = 0$." (In other words, the tangent line at the point $x = c$ is horizontal (or parallel to the x −axis)).

Assumptions for RT: (1) $f:[a,b] \to R$ is continuous, (2) $f:(a,b) \to R$ is differentiable, (3) $f(a) = f(b)$.

From RT to LMVT

(Idea) Rewrite $\frac{f(b)-f(a)}{b-a}=f'(d)$ (notice that we have a quotient on the left-hand side!) in the form of some function $p(x)$ satisfying (*) $p(a)=p(b)$ so that we can use Rolle's Theorem to get $p'(c)=0$.

Question: Can we do that?

Answer: Look at Cauchy Mean Value Theorem (which is more complicated than Lagrange's Mean Value Theorem) to get an idea. CMVT says: For some e between a & b

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(e)}{g'(e)}$$

Let's rewrite it in the form $\big(f(b)-f(a)\big)g'(e)-\big(g(b)-g(a)\big)f'(e)=0$

We think this way: If we consider the function

$$q(x)=\big(f(b)-f(a)\big)g(x)-\big(g(b)-g(a)\big)f(x)$$

Then perhaps it will satisfies the assumptions of the Rolle's Theorem.

That is, we have to check:

(i) $q(b)=?$ (ii) $q(a)=?$

It turns out that $q(a)=q(b)$. (Check it yourself !) Therefore the assumptions of Rolle's Theorem are satisfied. It follows that there exists e between a & b such that:

$$q'(e)=0$$

But $q'(x)=\big(f(b)-f(a)\big)g'(x)-\big(g(b)-g(a)\big)f'(x)$

So substituting $x=e$, we obtain

$$0=q'(e)=\big(f(b)-f(a)\big)g'(e)-\big(g(b)-g(a)\big)f'(e)$$

Rearranging, we obtain

$$\frac{\big(f(b)-f(a)\big)}{\big(g(b)-g(a)\big)}=\frac{f'(e)}{g'(e)}$$

This is what we wanted to prove.

Proof of LMVT

Coming back, we observe that the conclusion of LMVT, i.e.

$$\frac{f(b)-f(a)}{b-a}=f'(d)\ \exists d\ \text{ between } a\ \&\ b$$

can be understood as CMVT with $g(x)=x$. That is the following:

$$\frac{f(b)-f(a)}{b-a}=\frac{f'(d)}{\frac{d}{dx}x\Big|_{x=d}}=\frac{f'(d)}{1}$$

Now we repeat our idea used to prove the CMVT before and rewrite the fraction in the following form:

$$\big(f(b)-f(a)\big)g(x)-(b-a)f'(d)=0$$

This leads us to consider the function $p(x)$ defined by

$$p(x)=\big(f(b)-f(a)\big)x-(b-a)f(x)$$

Again, we try to check whether $p(a)=p(b)$ so that we can apply Rolle's Theorem.

Substituting $x=a$ and $x=b$ in $p(x)$ gives

$$p(a)=\big(f(b)-f(a)\big)a-(b-a)f(a)=\cdots$$

$$p(b)=\big(f(b)-f(a)\big)b-(b-a)f(b)=\cdots$$

So $p(a)=p(b)$. Therefore Rolle's Theorem says:

$$\exists d\ \text{ between } a\ \&\ b\ \text{ such that }\ p'(d)=0, \qquad i.e.$$

$$\big(f(b)-f(a)\big)-(b-a)f'(d)=0$$

Which is what we wanted to prove.

Relations to Taylor's Theorem

Two very simple cases of TT are:

$n=0$ case.

If we let $x = b,$ and $a = c,$ we obtain

$$f'(c^*) = \frac{f(x) - f(c)}{x - c}$$

I.e.

$$f(x) = f'(c^*)(x - c) + f(c)$$
$$= f(c) + f'(c^*)(x - c).$$

I.e.

$$f(x) = f(c) + \ Error\ term.$$

Error term = $\frac{f'(c^*)}{1!}(x - c)^1$

Remark: The $n = 0$ TT is just LMVT!

The $n = 1$ Taylor's Theorem

Can we improve on the $n = 0$ Taylore' Theorem (i.e. approximating the function $y = f(x)$ by the horizontal line $y = f(c)$?)

Answer: Yes. We can try next $f(x) = f(c) + f'(c)(x - c)$+Error,
Where this time, the "Error" term is of the form

$$Q(x - c)^2$$

where Q is some number which we want to find.

Why? Because when x is near to c, $x - c$ is a number whose absolute value is less than 1, so $(x - c)^2 < |x - c|$

Rearranging, we obtain

$$f(x) - f(c) - f'(c)(x - c) = Q(x - c)^2$$

Goal: Find a formula for the number "Q".

To see this: Rewrite the above equation as

$$\frac{f(x) - f(c) - f'(c)(x - c)}{(x - c)^2} = Q$$

Question: How to find this number Q?

Interpret the above as "CMVT", i.e.
Let $A(x) = f(x) - f(c) - f'(c)(x - c),$

$$B(x) = (x - c)^2$$

Then the above formula (LHS) is:

$$\frac{A(x) - A(c)}{B(x) - B(c)} = \frac{A'(d)}{B'(d)} = \frac{f'(d) - f'(c)1}{2(d - c)}$$

for some (or $\exists$) $d \in (c, x)$ or (x, c).
where d depends on c and x. Next, we use Cauchy Mean Value Theorem again to get

$$\frac{\left(\frac{1}{2}\right) f'(d) - (\frac{1}{2}) f'(c)}{d - c} = \left(\frac{1}{2}\right) f''(e) \ \ \exists e \ \ between \ d \,\&\, c.$$

Similarly, one can prove Taylor's Theorem for $n = 2,3,4, \cdots$ by repeated use of CMVT and LMVT.

For example when $n = 2,$ we get

$$\frac{f(x) - f(c) - f'(c)(x - c) - \frac{f''(c)}{2!}(x - c)^2}{(x - c)^3}$$

This expression, when we apply CMVT twice, then LMVT, will be equal to

$$\left(\frac{1}{3!}\right) f'''(\eta)$$