MATH1510E_H

Week 8

Topics

- Use of Taylor's Thm. to approximate, e.g. $\sin 1.5$
- Taylor series, Taylor polynomial

In short, Taylor's Theorem (i.e. T.T.) is a statement relating a given function $f(x)$ (the rule f is given, the variable x is given) to the following objects:

(i) the "approximating" polynomial $f(c)+f'(c)(x-c)+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n$

(those terms colored in red are "numbers")

together with an error term of the form

(ii) $\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$ (where ξ is a certain number between 0 and c).

The following is a typical example of using Taylor's Theorem.

Example of T.T.

Find an approximation of $\sin(1.5)$ with an error less than 0.01.

Solution: When we talk about error, the error can be "too much" or "too little", so the error 0.01 means ± 0.01.

Method: Use the formula

$$f(x)=f(c)+f'(c)(x-c)+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1},$$

and do the following:

(i) let $f(x)=\sin x$ and choosing $x=1.5$ and also choosing a convenient c

> e.g. $c=0$, because this leads to $f(c)=\sin 0=0$, $f'(c)=\cos 0=1$, $f''(c)=-\sin 0=0$, $f'''(c)=-\cos 0=-1,\cdots$, all easy to obtain without any computations!

(ii) Find n satsifying $\left|\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}\right|<0.01$ and by doing so,

(iii) obtain the approximation required, i.e. $f(c)+f'(c)(x-c)+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n=\sin 0+(\cos 0)\cdot(1.5-0)+(-\sin 0)\cdot\frac{(1.5-0)^2}{2!}+$

$$\cdots + \frac{\left.\frac{d^n \sin x}{dx^n}\right|_{x=0}}{n!} \cdot (1.5-0)^n$$ because $x = 1.5$ and $c = 0$.

The inequality $\left|\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}\right| < 0.01$ tells us what value of n is suitable.

Remark:

This inequality is difficult to handle, because we don't know the value of ξ, we only know that it lies between c and x.

Idea to obtain n, although we don't know the value of ξ.

To get n, we note first that $\left.\frac{d^n \sin x}{dx^n}\right|_{x=\xi} = \cos x,\ -\sin x,\ -\cos x$ and $\sin x$.

No matter what n, these functions give the estimate

$$\left|\frac{\left.\frac{d^{n+1} \sin x}{dx^{n+1}}\right|_{x=\xi}}{(n+1)!} \cdot (1.5-0)^{n+1}\right| < \left|\frac{1}{(n+1)!} \cdot (1.5-0)^{n+1}\right|$$

The number "1" in the numerator comes from the estimates, $|\sin x| \leq 1$, $|\cos x| \leq 1$.

Now we can do the following to estimate n. We can require n to satisfy the inequality

$$\left|\frac{1}{(n+1)!} \cdot (1.5-0)^{n+1}\right| < 0.01$$

Computing different values of $\frac{1}{(n+1)!} \cdot (1.5-0)^{n+1}$ we see that when $n \geq 7$, the inequality

$$\left|\frac{1}{(n+1)!} \cdot (1.5-0)^{n+1}\right| < 0.01$$

is satisfied.

Conclusion:

When $n \geq 7$, the error is less than 0.01. This means the following approximation of $\sin 1.5$:

$$\sin 0+\frac{\cos 0}{1!}x-\frac{\sin 0}{2!}x^2+\frac{-\cos 0}{3!}x^3+\frac{\sin 0}{4!}x^4+\frac{\cos 0}{5!}x^5+\frac{-\sin 0}{6!}x^6 +\frac{-\cos 0}{7!}x^7$$

(Note that the even power terms are all zero!)

Hence the required approximation of $\sin 1.5$ is:

$$1.5-\frac{1.5^3}{3!}+\frac{1}{5!}1.5^5-\frac{1}{7!}1.5^7$$
$$=0.997391$$

which differs from $\sin 1.5$ by an error less than 0.01.

Remarks:

1. Alternatively, one can think about the sequence
$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+\frac{(-1)^{k+1}x^{2k-1}}{(2k-1)!}+\cdots$$
in the following way: (i) x is the first term, (ii) $x^3/3!$ is the 2nd term, (iii) $\frac{(-1)^{k+1}x^{2k-1}}{(2k-1)!}$ is the k^{th} term.

   Then the error term is the $k+1^{th}$ term and has the form
$$\frac{f^{(2k)}(\xi)}{(2k+2)!}x^{2k}$$
2. Since the $(2k)^{th}$ derivatives of the function $\sin x$ is $\pm\sin x$, we obtain
$$\frac{f^{(2k)}(\xi)}{(2k+2)!}x^{2k}=\frac{\pm\sin\xi}{(2k+2)!}x^{2k}$$
3. Therefore we have
$$\left|\frac{f^{(2k)}(\xi)}{(2k+2)!}x^{2k}\right|=\left|\frac{\pm\sin\xi}{(2k+2)!}x^{2k}\right|\le\left|\frac{x^{2k}}{(2k+2)!}\right|=\left|\frac{1.5^{2k}}{(2k+2)!}\right|$$

Remark

The above method can also be used to estimate definite integrals which are difficult to compute, e.g. $\int_0^1 \sin\sqrt{x}\,dx$. (IDEA: Let $u=\sqrt{x}$ and use Taylor's Theorem for $\sin u$. Estimate also the remainder by using the inequalities $|\sin u|\le 1$ and $|\cos u|<1$).

Useful Terminologies for Taylor's Theorem

The following "terminologies" or "names" are useful in working with Taylor's Theorem.

Taylor polynomial of degree n

If a function (we assume that it has derivatives $f'(x), f''(x), \cdots$) satisfies

$$f(x) = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

We call the polynomial on the right-hand side (without the error term), i.e. the red-colored part, the "degree n Taylor polynomial of $f(x)$ at $x = c$". We denote it by $TP_n(x,c)$.

Examples:

1) Let $f(x) = e^x, c = 0$, then the degree n Taylor polynomial of $f(x)$ at $x = 0$ is $TP_n(x,0) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$

2) Let $f(x) = \frac{1}{1-x}, c = 0$, then the degree n Taylor polynomial of $f(x)$ at $x = 0$ is $TP_n(x,0) = 1 + x + x^2 + \cdots + x^n$

Taylor Series

If a function (we assume that it has derivatives $f'(x), f''(x), \cdots$) satisfies

$$f(x) = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

The error term, i.e. the term in yellow color, goes to zero as $n \to \infty$, then we have

$$f(x) = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

Or you can write the right-hand side in the form

$$\sum_{n=1}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

This "expression" is a called "Taylor series of the function $f(x)$ at the point $x = c$" (or "centered at $x = c$").

Some well-known Taylor Series

1) If $f(x) = e^x, c = 0$, then its Taylor series is $1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$

2) If $f(x) = \sin x$, $c = 0$, the its Taylor series is $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots +$

$(-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!} + \cdots$

3) If $f(x) = \cos x$, $c = 0$, the its Taylor series is $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots +$

$(-1)^n \frac{x^{2n}}{(2n)!} + \cdots$

4) If $f(x) = \ln(1 + x)$, $c = 0$, the its Taylor series is $x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots +$

$(-1)^{n+1} \frac{x^n}{n} + \cdots$

5) If $f(x) = \frac{1}{1-x}$, $c = 0$, the its Taylor series is $1 + x + x^2 + \cdots + x^n + \cdots$

6) If $f(x) = (1 + x)^\alpha$ the its Taylor series is $1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \cdots +$

$\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n + \cdots$

Remark:

In more advanced books, it will be shown that the "maximal" domain of the Taylor series in examples 1), 2) and 3) is the set of all real numbers (or denoted by $\mathbb{R}$), but the maximal domain of examples 4), 5) and 6) is the set of real numbers between 0 and 1.

Appendix

Supposing (for simplicity) that $x > 0$. The following outlines an argument to show that the Taylor series of $f(x) = e^x$ is $1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots$.

We need to show the following (where $0 < \xi < x$) :

$$\lim_{n\to\infty} \left| \frac{e^\xi}{(n+1)!} x^n \right| = 0.$$

Suppose we know $e < 3$, then all we need to show is the following limit:

$$\lim_{n\to\infty} \left| \frac{3^\xi}{(n+1)!} x^{n+1} \right| = 0$$

But now we know that $0 < \frac{3^{\xi}}{(n+1)!}x^{n+1} < \frac{3^{x}}{(n+1)!}x^{n+1}$, so we only need to show that

$$\lim_{n\to\infty} \frac{3^x}{(n+1)!}x^{n+1} = 0$$

This is seen as follows:

1. Fix x, then $x < m$ for some natural number m.
2. You can easily check that this limit is zero for small numbers such as $x = 1$ or $x = 2$.
3. For general m, the limit is zero follows from $3^m\left(\frac{m}{n+1}\right)\cdots\left(\frac{m}{m}\right)\left(\frac{m}{m-1}\right)\cdots\left(\frac{m}{1}\right)$ $< 3^m C\left(\frac{m}{n+1}\right)$ where $C = \left(\frac{m}{m}\right)\left(\frac{m}{m-1}\right)\cdots\left(\frac{m}{1}\right)$.
4. Letting $n \to \infty$ and using Sandwich Theorem gives the required limit.