MATH 1510H Notes

Week 6 (Day 2)
Week 7 (Day 1)

Topics covered

- Relationship between Extreme Value Theorem (EVT), Rolle's Theorem (RT), Lagrange's Mean Value Theorem (LMVT), Cauchy's Mean Value Theorem (CMVT), L'Hôpital's Rule (L'H), Taylor's Theorem (TT)
- The idea (which we will outline below) is: EVT $\Rightarrow$RT $\Rightarrow$ CMVT $\Rightarrow$ LMVT $\Rightarrow$ TT

  ("$\Rightarrow$" means "leads to" or "implies")
- Illustrated the Intermediate Value Theorem (IVT) via the example $x^{17} + 100x^2 + 107 = 0$.

Main Idea behind all the theorems like RT, LMVT, CMVT, L'H, TT are "existence theorems" (i.e. theorems which says "something" exists, but don't tell you how to find it).

The first of such existence theorems is:

Intermediate Value Theorem
Instead of writing down the theorem, we outline how it can used.
Suppose we asked:
Question
Show that the equation $x^{17} + 100x^2 + 107 = 0$ has a solution.

Answer:
We let $f(x) = x^{17} + 100x^2 + 107$. Then one sees that $f(-10) < 0$ and $f(0) > 0$. Now $f(x)$ is a continuous function, so the curve $y = f(x)$ must intersect the x −axis at some point between $x = -10$ and $x = 0$.
In other words, the equation $x^{17} + 100x^2 + 107 = 0$ must have a solution in $(-10,0)$.

This "method" is based on the

Theorem (Intermediate Value Theorem)

Assume $f: [a, b] \to R$ is continuous. Suppose that $f(a) < 0$ and $f(b) > 0$ (or $f(a) > 0$ and $f(b) < 0$).

Conclusion: The equation $f(x) = 0$ has at least one solution in (a, b).

Similar to this theorem is the Extreme Value Theorem

Theorem (Extreme Value Theorem)

Assume: $f: [a, b] \to R$ is continuous. Then there is an absolute maximum/minimum value in $[a, b]$. I.e.

$\exists c \in [a, b]$ such that $f(c) \geq f(x)$ $\forall x \in [a, b]$ (c is "absolute maximum point")

$\exists d \in [a, b]$ such that $f(d) \leq f(x)$ $\forall x \in [a, b]$ (c is "absolute maximum point").

Remark:

This EVT is the main engine behind the proofs of RT, LMVT, CMVT, TT.

Taylor's Theorem

Recall that Taylor's Theorem is one of the central results in Calculus. It tells us that we can "approximate" a function $f(x)$ by a "polynomial" in $(x - c)$.

It says: "Under some conditions (which we'll specify later),

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + E_n(x, c). \text{ "}$$

(Here the notation $f^{(n)}(c)$ means " n^{th} —derivative of the function $f(x)$ evaluated at $x = c$. " So it is a "number"!)

Remarks:

- The left-hand side (LHS) is a given function which we want to "approximate". The right-hand side (RHS) is a polynomial in $(x - c)$, together with an error term. The LHS is difficult to compute, if the function is complicated. But once we know "all" the derivatives of "$f(x)$", at the point $x = c$, we can compute the right-hand side i.e. $f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n$

  This is only an "approximation" because there is still an error term.

- The error term has a special form which is $\frac{f^{(n+1)}(d)}{(n+1)!}(x-c)^{n+1}$ (where d is some number lying between x and c.)
- The number c is something which we can choose. We usually choose some simple-to-compute numbers.

Taylor's Theorem and Lagrange's Mean Value Theorem

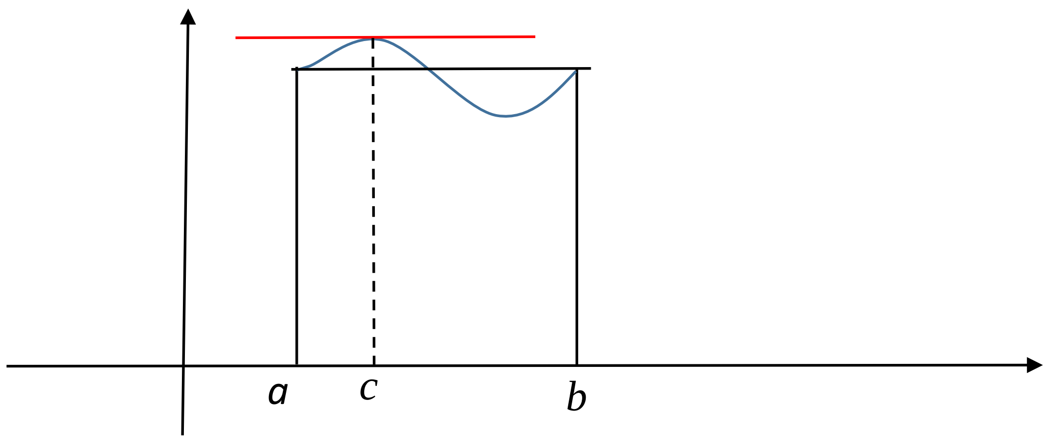
Taylor's Theorem (we may indicate how to prove it in the next lecture) is actually just a clever combination of Lagrange's Mean Value Theorem (LMVT) and Cauchy's Mean Value Theorem (CMVT).

But before everything, we have to describe what Lagrange's Mean Value Theorem & Cauchy's Mean Value Theorem are.

These two theorems, i.e. LMVT & CMVT are related to something known as the Rolle's Theorem, outlined below.

Rolle's Theorem

The following picture explains Rolle's Theorem:



Rolles' Theorem says: "If a function $f(x)$ satisfies (1), (2), (3) below, then $\exists c \in (a,b)$ such that $f'(c)=0$." (In other words, the tangent line at the point $x=c$ is horizontal (or parallel to the $x-$axis)).

Remark: Of course, there may be more than one such point!

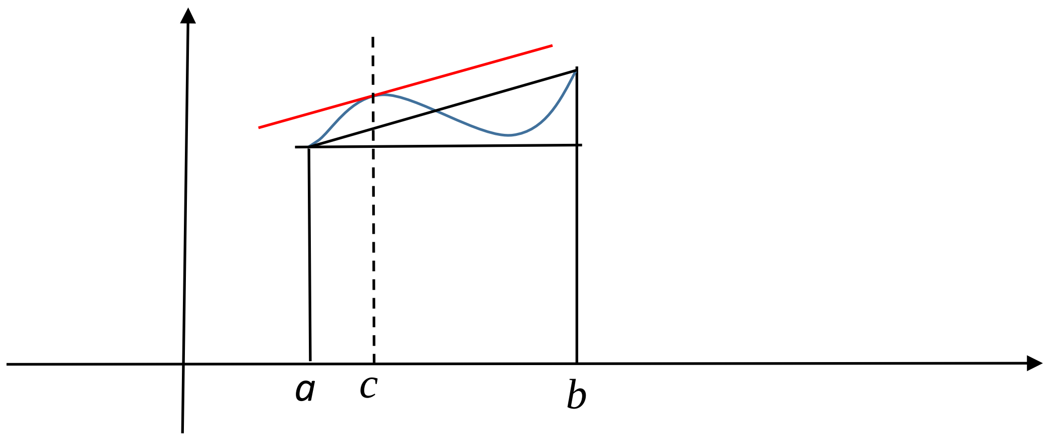
Assumptions for Rolle's Theorem:

1. $f(x)$ is differentiable in (a, b). (This assumption is needed, because in the conclusion, we have the expression $f'(c) = 0$)
2. $f(x)$ is continuous on [a,b] (This is "technical assumption")
3. $f(a) = f(b)$.

As mentioned above, Rolle's Theorem, when "rotated", gives the Lagrange's Mean Value Theorem.

Lagrange's Mean Value Theorem

It says: "If a function satisfies only (1) and (2) below, then $\exists d \in (a, b)$ such that: $f'(d) = \frac{f(b)-f(a)}{b-a}$."



Assumptions for LMVT

1. $f(x)$ is differentiable in (a, b). (This assumption is needed, because in the conclusion, we have the expression $f'(c) = \text{something.}$)
2. $f(x)$ is continuous on $[a, b]$. (This is again a "technical assumption").

Next, if we change slightly the Lagrange's Mean Value Theorem, we get

Cauchy's Mean Value Theorem

It says

Assumptions:

1. Let $f(x), g(x)$ be two differentiable functions in (a, b).
2. Let $f(x), g(x)$ be continuous on $[a, b]$.

3. Let $g'(x) \neq 0 \;\; \forall x \in (a,b)$. (This guarantees that the denominator is not zero.)
Then we have the Conclusion:

$$\exists \xi \in (a,b): \quad \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}$$

First Application of Cauchy's Mean Value Theorem – L'Hôpital's Rule

To explain what the L'Hôpital's Rule is, let's consider two examples.

Example 1

Find $\lim_{x \to 0} \frac{\sin x}{x}$.

This is a limit of the form $\frac{0}{0}$. The L'Hôpital's Rule says: "we can compute it via the procedure, $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\frac{d \sin x}{dx}}{\frac{dx}{dx}}$
(if the limit on the right-hand side exists).

Now the right-hand side is $\lim_{x \to 0} \frac{\frac{d \sin x}{dx}}{\frac{dx}{dx}} = \lim_{x \to 0} \frac{\cos x}{1} = 1$.

Hence the limit $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Question: What does the L'Hôpital's Rule say?
Answer: In the simplest case, it says the following:

Suppose $\lim_{x \to c} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, (where c is either a finite number or represents $\pm\infty$). Further, suppose that $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

The Proof:

(Idea) For simplicity, we assume that $f(c)=0, g(c)=0$, where c is a finite no (We will not prove the other cases).

The idea is to apply Cauchy's Mean Value Theorem and get

$$\frac{f(x)-f(c)}{g(x)-g(c)}=\frac{f'(d)}{g'(d)} \quad \exists d \text{ between } c \ \ \& \, x$$

("d between c & x" means "$c<d<x$ or $x<d<c$")

Now, remembering that $\frac{f(x)}{g(x)}=\frac{f(x)-f(c)}{g(x)-g(c)}$, (since $f(c)=0, g(c)=0$), we obtain from the two formulas above that

$$\frac{f(x)}{g(x)}=\frac{f'(d)}{g'(d)} \quad \exists d \text{ between } c \ \ \& \, x$$

Finally, we let $x \to c$ on the left-hand side to get

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(d)}{g'(d)} \quad \exists d \text{ between } c \ \ \& \, x \tag{1}$$

But "$c<d<x$ or $x<d<c$", so in either case, when $x \to c$, it follows that $d \to c$. This means the right-hand side of the above line becomes

$$\lim_{x\to c}\frac{f'(d)}{g'(d)}=\lim_{d\to c}\frac{f'(d)}{g'(d)}$$

We can now "rename" d to be x and obtain

$$\lim_{d\to c}\frac{f'(d)}{g'(d)}=\lim_{x\to c}\frac{f'(x)}{g'(x)} \tag{2}$$

Combining (1) and (2), we obtain

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$$

This is what wanted to prove.

Example 2

Find $\lim_{x\to 0}\left(\frac{1}{\sin^2 x}+\frac{\cos^2 x}{\sin^2 x}-\frac{2}{x^2}\right)$

This one is of the form $\infty-\infty$.

In cases like $\infty-\infty, \infty^\infty, 0^0$, we first rewrite them in the form

$\frac{\infty}{\infty}$ or $\frac{0}{0}$. After doing this, we use L'Hôpital's Rule.

In this example, $\frac{1}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} - \frac{2}{x^2} = \frac{x^2+x^2\cos^2 x - 2\sin^2 x}{x^2\sin^2 x}$

Now the limit of $\frac{x^2+x^2\cos^2 x - 2\sin^2 x}{x^2\sin^2 x}$ as $x \to 0$, is of the form $\frac{0}{0}$.

I.e. $\lim_{x\to 0}\frac{x^2+x^2\cos^2 x - 2\sin^2 x}{x^2\sin^2 x}$ is of the form $\frac{0}{0}$.

Applying several times L'Hôpital's Rule will give the answer $-\frac{1}{3}$.

(Alternative Method)

An easier method is to use the approximations $\sin x \approx x - \left(\frac{x^3}{3!}\right)$; $\cos x \approx 1 - \left(\frac{x^2}{2!}\right)$

Doing this, we obtain

$$\frac{x^2 + x^2\cos^2 x - 2\sin^2 x}{x^2\sin^2 x} \approx \frac{x^2 + x^2\left(1-\left(\frac{x^2}{2!}\right)\right)^2 - 2\left(x - \left(\frac{x^3}{3!}\right)\right)^2}{x^2\left(x-\left(\frac{x^3}{3!}\right)\right)^2}$$

$$= \frac{1 + \left(1-\left(\frac{x^2}{2!}\right)\right)^2 - 2\left(1 - \left(\frac{x^2}{3!}\right)\right)^2}{x^2\left(1-\left(\frac{x^2}{3!}\right)\right)^2}$$

$$= \frac{-1 + \left(\frac{x^2}{4}\right) + \left(\frac{2}{3}\right) - \left(\frac{x^2}{18}\right)}{1 - \left(\frac{x^2}{3}\right) + \left(\frac{x^4}{36}\right)}$$

Finally, we take the limit $x \to 0$ of the last expression, which gives

$$\lim_{x\to 0}\frac{-1 + \left(\frac{x^2}{4}\right) + \left(\frac{2}{3}\right) - \left(\frac{x^2}{18}\right)}{1 - \left(\frac{x^2}{3}\right) + \left(\frac{x^4}{36}\right)} = -\frac{1}{3}$$