THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010D&E (2016/17 Term 1)
University Mathematics
Tutorial 2 Solutions

Problems that may be demonstrated in class :

Assume we know the fact: $2 < e = \lim_{n\to\infty}(1+\frac{1}{n})^n$, $\lim_{n\to\infty}\sin\frac{1}{n} = 0$.

Q1. State whether the following sequence converges. Find the limit if it exists.

(a) $\frac{37(-n)^{2017}-(-n)^{689}}{141(-n)^{2017}+928(-n)^{64}}$; (b) $\sqrt[3]{2n^3+1} - \sqrt[3]{2n^3-n^2}$; (c) $(-1/2)^n$;

(d) $(1-\frac{1}{n+1})^n$; (e) $\sin\frac{n^2}{n+2} - \sin\frac{n^3-n-2}{n^2+2n}$; (f) $\frac{n^2}{\ln(n+1)}$;

(g) $\cos\frac{1}{n}$; (h) $\tan\frac{1}{n}$.

Q2. Let $\{a_n\}$ be a *harmonic sequence*, i.e. a sequence such that $a_n \neq 0$ for any $n \in \mathbb{N}$ and $1/a_n$ is an arithmetic sequence. Prove that $\{a_n\}$ converges.

Q3. Let $\{a_n\}$ be a sequence such that $a_n > 0$ for any $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = a > 0$. Use Sandwich Theorem to show that $\{\sqrt{a_n}\}$ converges and $\lim_{n\to\infty}\sqrt{a_n} = \sqrt{a}$.

Q4. Suppose $\{a_n\}$ is a sequence such that $a_1 \neq 0$ and $a_{n+1} = 2^{-1}(a_n + a_n^{-1})$ for any $n \in \mathbb{N}$. Does $\{a_n\}$ converge? If it does, find its limit.

Q5. Suppose for any $m \in \mathbb{N}$, we have a function $f_m(x) = x^2 - mx - 1, x \in \mathbb{R}$ and a sequence $\{a_{m,n}\}$ satisfying the recursive relation:

$$a_{m,n+1} = m + \frac{1}{a_{m,n}} \quad \text{for any } n \in \mathbb{N}, \quad a_{m,1} > 0.$$

(a) Fix $m \in \mathbb{N}$. Show that for any $n \in \mathbb{N}$, $a_{m,n} > 0$ and

$$f_m(a_{m,n+1}) = -\frac{f_m(a_{m,n})}{a_{m,n}^2} = \frac{a_{m,n+1} - a_{m,n}}{a_{m,n}}.$$

(b) Fix $m \in \mathbb{N}$. Show that $\{a_{m,2n-1}\}$ is monotonic decreasing and bounded below if $f_m(a_{m,1}) \geq 0$ and $\{a_{m,2n-1}\}$ is a monotonic increasing and bounded above if $f_m(a_{m,1}) < 0$.

(c) Fix $m \in \mathbb{N}$. Show that $\{a_{m,n}\}$ converges and find its limit a_m in terms of m.

(d) Evaluate $\lim_{m\to\infty} a_m$ and $\lim_{m\to\infty}(a_{m+1} - a_m)$.

Solution Q1. (a)

$$\lim_{n\to\infty}\frac{37(-n)^{2017}-(-n)^{689}}{141(-n)^{2017}+928(-n)^{64}} = \lim_{n\to\infty}\frac{-37n^{2017}+n^{689}}{-141n^{2017}+928n^{64}}$$
$$= \lim_{n\to\infty}\frac{-37+\frac{1}{n^{1382}}}{-141+\frac{928}{n^{1953}}} = \frac{-37}{-141} = \frac{37}{141}.$$

(b)

$$\begin{aligned}
&\lim_{n\to\infty}\left(\sqrt[3]{2n^3+1}-\sqrt[3]{2n^3-n^2}\right)\\
=&\lim_{n\to\infty}\frac{(2n^3+1)-(2n^3-n^2)}{\sqrt[3]{(2n^3+1)^2}+\sqrt[3]{(2n^3+1)(2n^3-n^2)}+\sqrt[3]{(2n^3-n^2)^2}}\\
=&\lim_{n\to\infty}\frac{1+n^{-2}}{\sqrt[3]{(2+n^{-3})^2}+\sqrt[3]{(2+n^{-3})(2-n^{-1})}+\sqrt[3]{(2-n^{-1})^2}}\\
=&\frac{1}{\sqrt[3]{2^2}+\sqrt[3]{2\cdot 2}+\sqrt[3]{2^2}}=\frac{1}{3\sqrt[3]{4}}.
\end{aligned}$$

(c) Since $0<1/2<1$, $\lim_{n\to\infty}1/2^n=0=\lim_{n\to\infty}-1/2^n$. Note that for any $n\in\mathbb{N}$, $-1/2^n\le(-1/2)^n\le 1/2^n$. By Sandwich Theorem, $\lim_{n\to\infty}(-1/2)^n=0$.

(d) Since $\lim_{n\to\infty}(1+\frac{1}{n})^n=e>0$,

$$\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\lim_{n\to\infty}\frac{1}{(1+\frac{1}{n})^n}=\frac{1}{e}.$$

(e) For any $n\in\mathbb{N}$,

$$\begin{aligned}
&\left|\sin\frac{n^2}{n+2}-\sin\frac{n^3-n-2}{n^2+2n}\right|\\
=&\left|2\sin\frac{1}{2}\left(\frac{n^2}{n+2}-\frac{n^3-n-2}{n^2+2n}\right)\cos\frac{1}{2}\left(\frac{n^2}{n+2}+\frac{n^3-n-2}{n^2+2n}\right)\right|\\
\le&\left|2\sin\frac{1}{2}\left(\frac{n^2}{n+2}-\frac{n^3-n-2}{n^2+2n}\right)\right|=2\sin\frac{1}{2n},\\
\therefore -2\sin\frac{1}{2n}\le&\sin\frac{n^2}{n+2}-\sin\frac{n^3-n-2}{n^2+2n}\le 2\sin\frac{1}{2n}.
\end{aligned}$$

We know that $\{\sin\frac{1}{2}\}$ is a subsequence of $\{\sin\frac{1}{n}\}$ which converges to 0. Thus $\lim_{n\to\infty}\sin\frac{1}{2n}=0$, therefore $\lim_{n\to\infty}2\sin\frac{1}{2n}=0=\lim_{n\to\infty}-2\sin\frac{1}{2n}$. By Sandwich Theorem, $\lim_{n\to\infty}(\sin\frac{n^2}{n+2}-\sin\frac{n^3-n-2}{n^2+2n})=0$.

(f) Note that $1+1=2<e^1$. Assume $k+1<e^k$ for some $k\in\mathbb{N}$. Then

$$k+2\le e^k+1\le 2e^k<e^{k+1}.$$

By mathematical induction, $n+1<e^n$ for any $n\in\mathbb{N}$. Then for any $n\in\mathbb{N}$,

$$\ln(n+1)<n \quad\text{and thus}\quad n<\frac{n^2}{\ln(n+1)}.$$

Because $\lim_{n\to\infty}n=+\infty$, $\lim_{n\to\infty}n^2/\ln(n+1)=+\infty$.

(g) As in (e), since $\lim_{n\to\infty}\sin\frac{1}{n}=0$ and $\{\sin\frac{1}{2n}\}$ is a subsequence of $\{\sin\frac{1}{n}\}$, $\lim_{n\to\infty}\sin\frac{1}{2n}=0$. Then

$$\lim_{n\to\infty}\cos\frac{1}{n}=\lim_{n\to\infty}\left(1-\sin^2\frac{1}{2n}\right)=1-0^2=1.$$

(h) As $\lim_{n\to\infty}\sin\frac{1}{n}=0$ and $\lim_{n\to\infty}\cos\frac{1}{n}=1\ne 0$,

$$\lim_{n\to\infty}\tan\frac{1}{n}=\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\cos\frac{1}{n}}=\frac{0}{1}=0.$$

Q2. There exist real numbers a and d such that $1/a_n = a + nd$ for any $n \in \mathbb{N}$. If $d = 0$, then $a = 1/a_n \neq 0$ for any $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1/a = 1/a$. If $d \neq 0$, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{a+nd} = \lim_{n\to\infty} \frac{n^{-1}}{an^{-1}+d} = 0$. $\{a_n\}$ converges.

Q3. For any $n \in \mathbb{N}$,

$$0 \leq \left|\sqrt{a_n} - \sqrt{a}\right| = \left|\frac{a_n - a}{\sqrt{a_n} + \sqrt{a}}\right| = \frac{|a_n - a|}{\sqrt{a_n} + \sqrt{a}} \leq \frac{|a_n - a|}{\sqrt{a}},$$
$$\therefore -\frac{|a_n - a|}{\sqrt{a}} \leq \sqrt{a_n} - a \leq \frac{|a_n - a|}{\sqrt{a}}.$$

We see that $\lim_{n\to\infty}(a_n - a) = a - a = 0$, whence $\lim_{n\to\infty} \frac{|a_n - a|}{\sqrt{a}} = \frac{0}{\sqrt{a}} = 0$ and $\lim_{n\to\infty} \frac{-|a_n-a|}{\sqrt{a}} = 0$. By Sandwich Theorem, $\lim_{n\to\infty}(\sqrt{a_n} - a) = 0$, implying that $\lim_{n\to\infty} \sqrt{a_n} = \lim_{n\to\infty}(\sqrt{a_n} - \sqrt{a}) + \sqrt{a} = \sqrt{a}$.

Q4. We consider the cases when $a_1 > 0$ and when $a_1 < 0$ separately.
Case (1): $a_1 > 0$. Assume $a_k > 0$ for some $k \in \mathbb{N}$. Then $a_{k+1} = 2^{-1}(a_k + a_k^{-1}) > 0$. By mathematical induction, $a_n > 0$ for any $n \in \mathbb{N}$. Observe that for any $n \in \mathbb{N}$,

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{1}{a_n}\right) = 1 + \frac{1}{2}\left(a_n - 2 + \frac{1}{a_n}\right) = 1 + \frac{1}{2}\left(\sqrt{a_n} - \frac{1}{\sqrt{a_n}}\right)^2 \geq 1,$$
$$\therefore a_{n+2} = \frac{1}{2}\left(a_{n+1} + \frac{1}{a_{n+1}}\right) = a_{n+1} + \frac{1 - a_{n+1}^2}{2a_{n+1}} \leq a_{n+1} + \frac{1 - 1^2}{2a_{n+1}} = a_{n+1}.$$

Hence $\{a_{n+1}\}$ is monotonic decreasing and bounded below by 1. By Monotone Convergence Theorem, $\{a_{n+1}\}$ converges. Let $a = \lim_{n\to\infty} a_{n+1}$. Then $a \geq 1$ and

$$2a^2 = \lim_{n\to\infty} 2a_n a_{n+1} = \lim_{n\to\infty} (a_n^2 + 1) = a^2 + 1,$$
$$(a+1)(a-1) = a^2 - 1 = 0,$$
$$\therefore a = 1 \text{ or } -1 \text{ (rejected)}.$$

Case (2): $a_1 < 0$. Define $b_n = -a_n$ for any $n \in \mathbb{N}$. Then $b_1 = -a_1 > 0$ and $b_n = -a_n = -2^{-1}(a_n + a_n^{-1}) = 2^{-1}(b_n + b_n^{-1})$ for any $n \in \mathbb{N}$. Applying case (1), $\lim_{n\to\infty} b_n = 1$, whence $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-b_n) = -1$.
Combining the two cases, we conclude that the sequence $\{a_n\}$ converges and

$$\lim_{n\to\infty} a_n = \begin{cases} 1, & \text{if } a_1 > 0; \\ -1, & \text{if } a_1 < 0. \end{cases}$$

Q5. (a) By assumption, $a_{m,1} > 0$. Assume $a_{m,k} > 0$ for some $k \in \mathbb{N}$. Then we have $a_{m,k+1} = m + \frac{1}{a_{m,k}} > m > 0$. By mathematical induction, $a_{m,n} > 0$ for any $n \in \mathbb{N}$. For any $n \in \mathbb{N}$,

$$a_{m,n}^2 f_m(a_{m,n+1}) = a_{m,n}^2 \left[\left(m + \frac{1}{a_{m,n}}\right)^2 - m\left(m + \frac{1}{a_{m,n}}\right) - 1\right]$$
$$= m a_{m,n} + 1 - a_{m,n}^2 = -f_m(a_{m,n}),$$
$$\therefore \frac{a_{m,n+1} - a_{m,n}}{a_{m,n}} = \frac{m a_{m,n} + 1 - a_{m,n}^2}{a_{m,n}^2} = -\frac{f_m(a_{m,n})}{a_{m,n}^2} = f_m(a_{m,n+1}).$$

(b) First notice that for any $n \in \mathbb{N}$,

$$\begin{aligned} a_{m,n+2} - a_{m,n} &= (a_{m,n+2} - a_{m,n+1}) + (a_{m,n+1} - a_{m,n}) \\ &= a_{m,n+1} f_m(a_{m,n+2}) + a_{m,n} f_m(a_{m,n+1}) \\ &= a_{m,n+1}(1 - a_{m,n} a_{m,n+1}) f_m(a_{m,n+2}) \\ &= -m a_{m,n} a_{m,n+1} f_m(a_{m,n+2}) \\ &= \frac{m a_{m,n} f_m(a_{m,n+1})}{a_{m,n+1}} = -\frac{m f_m(a_{m,n})}{a_{m,n} a_{m,n+1}}, \\ \therefore a_{m,n+4} - a_{m,n+2} &= -\frac{m f_m(a_{m,n+2})}{a_{m,n+2} a_{m,n+3}} = \frac{a_{m,n+2} - a_{m,n}}{a_{m,n} a_{m,n+1} a_{m,n+2} a_{m,n+3}}. \end{aligned}$$

Suppose $f_m(a_{m,1}) \geq 0$. Then $a_{m,3} - a_{m,1} = -\frac{m f_m(a_{m,1})}{a_{m,1} a_{m,3}} \leq 0$.
Assume $a_{m,2k+1} - a_{2k-1} \leq 0$ for some $k \in \mathbb{N}$. Then

$$a_{m,2k+3} - a_{m,2k+1} = \frac{a_{m,2k+1} - a_{m,2k-1}}{a_{m,2k-1} a_{m,2k} a_{m,2k+1} a_{m,2k+2}} \leq 0.$$

By mathematical induction, $\{a_{m,2n-1}\}$ is monotonic decreasing. Clearly, the sequence $\{a_{m,2n-1}\}$ is bounded below by 0.
Suppose $f_m(a_{m,1}) < 0$. Then $a_{m,3} - a_{m,1} = -\frac{m f_m(a_{m,1})}{a_{m,1} a_{m,3}} > 0$.
Assume $a_{m,2k+1} - a_{2k-1} > 0$ for some $k \in \mathbb{N}$. Then

$$a_{m,2k+3} - a_{m,2k+1} = \frac{a_{m,2k+1} - a_{m,2k-1}}{a_{m,2k-1} a_{m,2k} a_{m,2k+1} a_{m,2k+2}} > 0.$$

By mathematical induction, $\{a_{m,2n-1}\}$ is monotonic increasing. Consider any $n \in \mathbb{N}$. We have $f_m(a_{m,2n-1}) = -m^{-1} a_{m,2n-1} a_{m,2n} (a_{m,2n+1} - a_{m,2n-1}) \leq 0$. For any real number $x > 2m$, $f_m(x) = x(x-m) - 1 \geq 2m(2m-m) - 1 = 2m^2 - 1 > 0$. Hence $a_{m,2n-1} \leq 2m$. The sequence $\{a_{m,2n-1}\}$ is bounded above by $2m$.

(c) By part (b) and Monotone Convergence Theorem, $\{a_{m,2n-1}\}$ converges. Let $b_{m,n} = a_{m,n+1}$ for any $n \in \mathbb{N}$. Since $b_{m,1} = a_{m,2} > 0$ and $b_{m,n+1} = m + \frac{1}{b_{m,n}}$ for any $n \in \mathbb{N}$, $\{b_{m,2n-1}\}$ converges and so does $\{a_{m,2n}\}$.

$$\begin{aligned} \lim_{n\to\infty} (a_{m,2n} - a_{m,2n-1}) &= \lim_{n\to\infty} a_{m,2n-1} f_m(a_{m,2n}) \\ &= \lim_{n\to\infty} m^{-1} a_{m,2n-1} a_{m,2n} a_{m,2n+1} (a_{m,2n} - a_{m,2n+2}) = 0, \\ \therefore \lim_{n\to\infty} a_{m,2n} &= \lim_{n\to\infty} a_{m,2n-1}. \end{aligned}$$

Therefore, $\{a_{m,n}\}$ converges. Let $a_m = \lim_{n\to\infty} a_{m,n}$. Since $a_{m,n} \geq 0$ for any $n \in \mathbb{N}$, we have $a_m \geq 0$.

$$\begin{aligned} a_m^2 = \lim_{n\to\infty} a_{m,n} a_{m,n+1} &= \lim_{n\to\infty} (m a_{m,n} + 1) = m a_m + 1, \\ f_m(a_m) &= a_m^2 - m a_m + 1 = 0, \\ a_m &= \frac{m + \sqrt{m^2+4}}{2} \text{ or } \frac{m - \sqrt{m^2+4}}{2} \text{ (rejected).} \end{aligned}$$

(d) For any $m \in \mathbb{N}$, $a_m \geq 2^{-1}(m + \sqrt{m^2}) = m$. Since $\lim_{m\to\infty} m = +\infty$, $\lim_{m\to\infty} a_m = +\infty$.

$$\begin{aligned}
\lim_{m\to\infty}(a_{m+1} - a_m) &= \lim_{m\to\infty} \frac{m + 1 + \sqrt{(m+1)^2 + 4} - m - \sqrt{m^2 + 4}}{2} \\
&= \lim_{m\to\infty} \left(\frac{1}{2} + \frac{((m+1)^2 + 4) - (m^2 + 4)}{2(\sqrt{(m+1)^2 + 4} + \sqrt{m^2 + 4})} \right) \\
&= \lim_{m\to\infty} \left(\frac{1}{2} + \frac{2m + 1}{2(\sqrt{(m+1)^2 + 4} + \sqrt{m^2 + 4})} \right) \\
&= \lim_{m\to\infty} \left(\frac{1}{2} + \frac{2 + \frac{1}{m}}{2\left(\sqrt{(1 + \frac{1}{m})^2 + \frac{4}{m^2}} + \sqrt{1 + \frac{4}{m^2}}\right)} \right) \\
&= \frac{1}{2} + \frac{2}{2(1+1)} = 1.
\end{aligned}$$