**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**MATH1010D&E (2016/17 Term 1)**
**University Mathematics**
**Tutorial 9**

**Theorem (Operations of integral)**

Let $f, g$ be integrable on $[a,b]$, $a<c<b$ and $k$ be constants. Then

1. $\displaystyle\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
2. $\displaystyle\int_a^b kf(x)dx = k\int_a^b f(x)dx$
3. $\displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
4. $\displaystyle\int_b^a f(x)dx = -\int_a^b f(x)dx$

**Theorem (Fundamental theorem of calculus)**

Let $f$ be continuous on $[a,b]$.

1. *First part*: Let $F:[a,b]\to\mathbb{R}$ be the function defined by

$$F(x) = \int_a^x f(t)dt$$

Then $F$ is continuous on $[a,b]$, differentiable on $(a,b)$ and

$$F'(x) = f(x) \quad \forall\, x \in (a,b)$$

In other words, we have

$$\frac{d}{dx}\int_a^x f(t)dt = f(x) \quad \forall\, x \in (a,b)$$

2. *Second part*: Let $F(x)$ be a primitive function of $f(x)$. Then

$$\int_a^b f(x)dx = F(b) - F(a)$$

**Problems that may be demonstrated in class :**

Q1. Compute the following definite integrals:
(a) $\int_0^3 (x-1)(x+2)\,dx$ (b) $\int_0^3 x\lfloor x \rfloor dx$ (c) $\int_0^{\pi/2} \sin x \cos^4 x dx$ (d) $\int_1^e \frac{\ln x}{x} dx$

(e) $\int_2^4 \sqrt{16-x^2}\,dx$

Here $\lfloor x \rfloor$ is the greatest integer that is less than or equal to $x$.

Q2. (a) Let $f$ be continuous on $[0,1]$. Prove that $\int_0^\pi x f(\sin x) dx = \frac{\pi}{2} \int_0^\pi f(\sin x) dx$
(b) Evaluate $\int_0^\pi \frac{x \sin x}{1+\cos^2 x} dx$

Q3. Suppose $a > 0$ and that $f$ is continous on $\mathbb{R}$.
(a) If $f$ is even, show that $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$
(b) If $f$ is odd, show that $\int_{-a}^a f(x)dx = 0$
(c) Show that $\int_{-a}^a f(x^2)dx = 2\int_0^a f(x^2)dx$

Q4. Suppose that $f$ is continuous on $[a,b]$, $f(x) \geq 0$ for all $x \in [a,b]$ and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$.

Q5. Compute $F'(x)$ if $F(x)$ equals
(a) $\int_1^x e^{t^2}\,dt$ (b) $\int_1^{x^2} e^{t^2}\,dt$ (c) $\int_0^{3x} \tan(t^2)dt$ (d) $\int_{-x}^{x^2+3} \arctan t\,dt$

Q6. Let $f$ be a continuous function on $\mathbb{R}$ and $F$ be a primitive function of $f$. Let $a, b \in \mathbb{R}$ and $a < b$. Show that there exists $c \in (a,b)$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$