THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5000 Analysis I 2015-2016
Problem Set 7: integration

1. Let $f : [0, 1] \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 0 & \text{if} \quad x = 0; \\ 1 & \text{if} \quad 0 < x \leq 1. \end{cases}$$

Show that f is Riemann / Darboux integrable and the integral is 1.

(Remark: Without using the equivalence of Riemann integrable and Darboux integrable.)

2. Let $f : [0, 1] \to \mathbb{R}$ be a function defined by $f(x) = x$.

Show that f is Riemann / Darboux integrable and the integral is $\frac{1}{2}$.

3. Let $f : [0, 1] \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q}; \\ 1 & \text{if} \quad x \in \mathbb{R} \backslash \mathbb{Q}. \end{cases}$$

Show that f is Riemann / Darboux integrable and the integral is 1.

4. Let $f : [0, 1] \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some natual number } n; \\ 0 & \text{otherwise.} \end{cases}$$

Is f integrable on $[0, 1]$? Why?

5. (a) Let $f : [a, b] \to \mathbb{R}$ be a function which is integrable on $[a, b]$. Show directly that $|f|$ is integrable.

(b) Let $f, g : [a, b] \to \mathbb{R}$ be functions which are integrable on $[a, b]$. Define $H : [a, b] \to \mathbb{R}$ by $H(x) = \max\{f(x), g(x)\}$ for all $x \in [a, b]$. Show that H is integrable on $[a, b]$.

(Hint: Show that $H(x) = \dfrac{f(x) + g(x) + |f(x) - g(x)|}{2}$ and recall the linearity of integrals.)

6. Let $f, g : [a, b] \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$. Suppose that $(P, \vec{c})$ is a tagged partition of $[a, b]$, show that

$$S(\alpha f + \beta g, P, \vec{c}) = \alpha S(f, P, \vec{c}) + \beta S(g, P, \vec{c}).$$

Suppose that $f, g : [a, b] \to \mathbb{R}$ are Riemann integrable, by using the above result, show that $\alpha f + \beta g$ are Riemann integrable on $[a, b]$.

7. (a) Let $f : [a, b] \to \mathbb{R}$ be a bounded function such that $f(x) = 0$ except possibly for a finite number of points on $[a, b]$. Prove that f is integrable on $[a, b]$ and $\displaystyle\int_a^b f = 0$.

(b) Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions such that $f(x) = g(x)$ except possibly for a finite number of points on $[a, b]$. Prove that f is integrable on $[a, b]$ if and only if g is integrable on $[a, b]$.

8. Let $f : [a, b] \to \mathbb{R}$ be integrable, and let P be an even partition of $[a, b]$ given by

$$P = \{x_0, x_1, \ldots, x_n\}, \qquad x_i = a + \frac{(b-a)i}{n} \text{ for } i = 1, 2, \ldots, n.$$

Define the trapezoidal rule by

$$T_n(P, f) = \frac{b-a}{n} \sum_{i=1}^{\infty} \left(\frac{f(x_{i-1} + f(x_i)}{2} \right).$$

Show that $\lim_{n\to\infty} T_n(P, f) = \int_a^b f$.

9. (Extension of the First Fundamental Theorem of Calculus) Let $f : [a, b] \to \mathbb{R}$ be integrable. Suppose that the function $F : [a, b] \to \mathbb{R}$ is continuous, that $F : (a, b) \to \mathbb{R}$ is differentiable, and that $F'(x) = f(x)$ for all $x \in (a, b)$ except possibly finitely many points. Prove that

$$\int_a^b f = F(b) - F(a).$$

10. Let $f : [a, b] \to \mathbb{R}$ be an integrable function such that $f(x) \geq m$ for all $x \in [a, b]$ for some $m > 0$. Suppose that $1/f : [a, b] \to \mathbb{R}$ be the reciprocal function.

(a) Let P be a partition of $[a, b]$. Show that

$$U(1/f, P) - L(1/f, P) \leq \frac{1}{m^2}[U(f, P) - L(f, P)].$$

(b) Hence, prove that $1/f$ is integrable on $[a, b]$.