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Not to be taken away

香 港 中 文 大 學

The Chinese University of Hong Kong

二〇一五至一六年度 上學期科目考試

Course Examination 1st Term, 2015-16

科目編號及名稱 Course Code & Title : **MMAT5000 Analysis I**

時間 Time allowed : **2** 小時 hours **30** 分鐘 minutes

學號 Student I.D. No. : ____________ 座號 Seat No. : ____________

Time allowed: 2 hr 30 min

Total points: 50

1. Let $f : (0, \infty)$ be a function defined by $f(x) = \dfrac{1}{x^2}$.

   (a) (i) State without proof the Mean Value Theorem.

   (ii) Show that f is uniformly continuous on $[1, \infty)$.
   (Hint: Show that f is a Lipschitz function on $[1, \infty)$.)

   (b) Show that f is not uniformly continuous on $(0, \infty)$.

   (10 Points)

2. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Prove that for any $\epsilon > 0$, there exists a continuous piecewise linear function $g : [a, b] \to \mathbb{R}$ such that $|f(x) - g(x)| < \epsilon$ for all $x \in [a, b]$.

   (Hint: You may use the fact that f is uniformly continuous on $[a, b]$.)

   (6 Points)

3. Let $f : [0, 2] \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 \le x < 1; \\ 2 & \text{if} \quad 1 \le x < 2; \\ 3 & \text{if} \quad x = 2. \end{cases}$$

   Prove that f is integrable on $[0, 2]$.

   (5 Points)

4. Let $f : [a, b] \to \mathbb{R}$ be integrable, and let P be an even partition of $[a, b]$ given by

$$P = \{x_0, x_1, \ldots, x_n\}, \qquad x_i = a + \frac{(b-a)i}{n} \text{ for } i = 1, 2, \ldots, n.$$

Let $h_n = \dfrac{b-a}{n}$ and define the mid-point rule by

$$M_n(P, f) = h_n \sum_{i=1}^{n} f(a + (i - \frac{1}{2})h_n).$$

Show that $\lim_{n\to\infty} M_n(P, f) = \int_a^b f$.

(5 Points)

5. Let $f : [a, b] \to \mathbb{R}$ be a nonnegative function.

Suppose that f is integrable on $[a, b]$ and $\int_a^b f = 0$.

(a) Give (without proof) a counterexample that f is not the zero function.

(b) If it is further known that f is continuous on $[a, b]$, must f be the zero function? Why?

(8 Points)

6. Let $d : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a function defined by $\left|\ln\left(\frac{y}{x}\right)\right|$.

Prove that d defines a metric on $\mathbb{R}$.

(8 Points)

7. Let X be the vector space of bounded sequences in $\mathbb{R}$ and $\|\cdot\| : X \to \mathbb{R}$ be a function defined

$$\|\{x_n\}\| = \sup\{|x_1|, |x_2|, \cdots\},$$

where $\{x_n\}$ is a sequence in $\mathbb{R}$.

Prove that $\|\cdot\|$ defines a norm on X.

(8 Points)

8. (Bouns Question) Let $f : [0, 1] \to \mathbb{R}$ be a function defined by the following:

- $f(0) = f(1) = 1$;
- If $0 < x < 1$ and x is irrational, then $f(x) = 0$;
- If $0 < x < 1$ and $x = \dfrac{m}{n}$ where m and n are natural numbers with $\gcd(m, n) = 1$, then $f(x) = \dfrac{1}{n}$.

(a) Prove that f is discontinuous at every rational number in $[0, 1]$.

(b) Prove that f is continuous at every irrational number in $[0, 1]$.

(c) Prove that f is integrable on $[0, 1]$.

(10 Points)