THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5000 Analysis I (Fall 2015)
Makeup Quiz 1

1. (a) Suppose that A and B are nonempty subsets of $\mathbb{R}$ such that $a < b$ for all $a \in A$ and $b \in B$. Prove that $\sup A \leq \inf B$.

   (b) Suppose that S is a nonempty subset of $\mathbb{R}$ which is bounded above. Define

   $$-S = \{-s \in \mathbb{R} : s \in S\}.$$

   Prove that $\inf(-S)$ exists and $\inf(-S) = -\sup S$.

   (c) Give an example of nonempty subsets $A_n \subset \mathbb{R}$, $n \in \mathbb{N}$ such that $A_i \cap A_j$ is nonempty for any $i \neq j$, but $\bigcap_{n=1}^{\infty} A_n$ is empty.

2. Suppose that $\{x_n\}$ is a sequence of real numbers such that $0 < x_n < 1$ for all $n \in \mathbb{N}$. Define $A = \bigcup_{n=1}^{\infty} (\frac{x_n}{n}, x_n)$.

   (a) Show that A is bounded.

   (b) Show that $\inf A = 0$ and $\sup A = \sup\{x_n : n \in \mathbb{N}\}$.

3. Let $S = \left\{ \dfrac{m}{n + \sqrt{2}} : m, n \in \mathbb{Z} \right\}$.
   Show that S is a dense subset of $\mathbb{R}$, i.e. for all real numbers $x, y \in \mathbb{R}$ with $x < y$, there exists $s \in S$ such that $x < s < y$.

4. (a) State without proof of the **Nested Interval Property**.

   (b) Suppose I_n, $n \in \mathbb{N}$ is a nested sequence of closed bounded intervals.

      (i) Prove that I_{3n} is a nested sequence of closed bounded intervals.

      (ii) Suppose that there exists **unique** $\xi \in \mathbb{R}$ such that $\xi \in I_{3n}$ for all $n \in \mathbb{N}$. Show that there exists **unique** $\xi \in I_n$ for all $n \in \mathbb{N}$.