§ 8 Towards Topology and Functional Analysis

Recall:

A sequence $\{x_n\}$ in $\mathbb{R}$ is said to converge to $x \in \mathbb{R}$, if

For all $\varepsilon > 0$, there exists $K(\varepsilon) \in \mathbb{N}$ such that for all $n \geq K(\varepsilon)$, we have $|x_n - x| < \varepsilon$

i.e. $x_n \in \underbrace{V_\varepsilon(x)}_{\text{open subset containing } x}$

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $c \in A$. We say that f is continuous at c if

given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in A$ with $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

Another formulation:

given any $\varepsilon > 0$, there exists $\delta > 0$ such that $f(\underbrace{A \cap V_\delta(c)}_{\text{open subset in } A \text{ containing } c}) \subseteq \underbrace{V_\varepsilon(f(c))}_{\text{open subset in } \mathbb{R} \text{ containing } f(c)}$

⋮ More examples

Key issue:

X: set (space)

What we need to specify is collection of open subsets of X (with certain properties).

8.1 Topological Spaces

Definition:

A topology on a set X is a collection $\mathcal{T}$ of open subsets of X such that:

1) $\phi, X \in \mathcal{T}$
2) [+]Any union of elements in $\mathcal{T}$ is in $\mathcal{T}$.
3) Finite intersection of elements in $\mathcal{T}$ is in $\mathcal{T}$.

A topological space is an ordered pair $(X, \mathcal{T})$ consisting a set X and a topology $\mathcal{T}$ on X. $U \in \mathcal{T}$ is said to be an open subset of X (with respect to $\mathcal{T}$).

[+] Let I be an index set and each $\alpha \in I$ corresponds to a subset $U_\alpha \subseteq X$.

Then we can define $\bigcup_{\alpha \in I} U_\alpha = \{x \in X : x \in U_\alpha \text{ for some } \alpha \in I\}$

Finite (countable) union means the index set I is finite (countable).

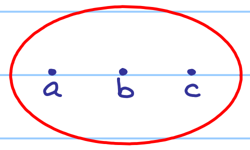
Any union means there is no restriction on I.

Similar terminologies we can define for intersection of subsets in X.

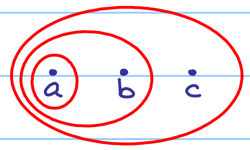
e.g. Let $U_\alpha = (\alpha, +\infty)$, $I = (0,1) \cap \mathbb{R} \setminus \mathbb{Q}$, then $\bigcup_{\alpha \in I} U_\alpha = (0, +\infty)$ and $\bigcap_{\alpha \in I} U_\alpha = [1, +\infty)$.

Example :

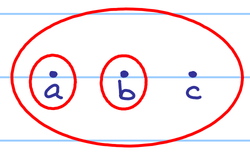
Let $X = \{a, b, c\}$



$\mathcal{T}_1 = \{\phi, X\}$



$\mathcal{T}_2 = \{\phi, \{a\}, \{a,b\}, X\}$



$\mathcal{T}_3 = \{\phi, \{a\}, \{b\}, X\}$

$\mathcal{T}_1, \mathcal{T}_2$ defines two different topologies on X,
but $\mathcal{T}_3$ does not define a topology on X.
(since $\{a\}, \{b\} \in \mathcal{T}_3$, but $\{a\} \cup \{b\} = \{a,b\} \notin \mathcal{T}$)

Exercise :

Can we define topologies other than $\mathcal{T}_1$ and $\mathcal{T}_2$ on X ?

Example :

Let X be a set and let $\mathcal{T}$ be the collection of subsets of X such that
$U \in \mathcal{T}$ if and only if $X \setminus U$ is either X or a finite set.

Exercise :

Check $\mathcal{T}$ defines a topology on X (which is called the finite complement topology)

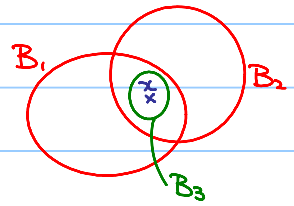
However, it is usually hard to describe the entire collection $\mathcal{T}$ of open subsets.
Therefore, we try to describe a smaller collection of subsets of X and describe
$\mathcal{T}$ in terms of that (describing building blocks !)

Definition:

Let X be a set, a basis on X is a collection $\mathcal{B}$ of subsets of X such that

1) For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.

2) If $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.



Let $\mathcal{B}$ be a basis.

We define $\mathcal{T}$ to be the collection of subsets of $U \subseteq X$ such that for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Exercise:

Check $\mathcal{T}$ defines a topology on X which is called the topology generated by $\mathcal{B}$.

Remark: $B \in \mathcal{T}$ for all $B \in \mathcal{B}$.

Examples:

1) Let $\mathcal{B} = \{(a,b) . a,b \in \mathbb{R} \text{ with } a<b\}$.

   The topology generated by $\mathcal{B}$ is called the standard topology on $\mathbb{R}$.

2) Let $\mathcal{B}' = \{[a,b) . a,b \in \mathbb{R} \text{ with } a<b\}$.

   The topology generated by $\mathcal{B}'$ is called the lower limit topology on $\mathbb{R}$.

Exercise:

Check $\mathcal{B}$ and $\mathcal{B}'$ are bases of $\mathbb{R}$.

Exercises:

1) Suppose that $(X, \mathcal{T})$ is a topological space and $A \subseteq X$.

   Let $\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}\}$, prove that $\mathcal{T}_A$ defines a topology on A which is called the induced topology on A from X.

2) Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces.

   Let $\mathcal{T}_{X \times Y} = \{U \times V : U \in \mathcal{T}_X \text{ and } V \in \mathcal{T}_Y\}$, prove that $\mathcal{T}_{X \times Y}$ defines a topology on $X \times Y$. which is called the product topology.

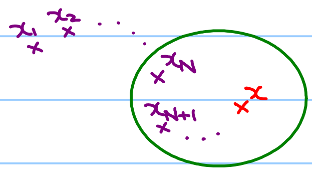
Remarks:

1) We have a standard topology on $\mathbb{R}$. The product of this topology with itself is called the standard topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

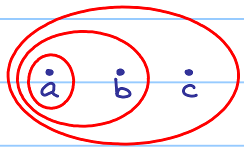
2) Let $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\} \subseteq \mathbb{R}^2$.
S^1 is a topological space with topology induced from $\mathbb{R}^2$.

Definition:

Let $(X, \mathcal{T})$ be a topological space and let $\{x_n\} \subseteq X$ be a sequence.
$\{x_n\}$ is said to converge to $x \in X$, if for all $U \in \mathcal{T}$ with $x \in U$, there exists $n \geq N$ such that for all $n \geq N$, $x_n \in U$.



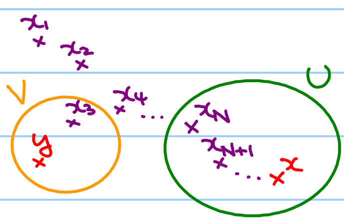
However, consider $X = \{a,b,c\}$, $\mathcal{T} = \{\phi, \{a\}, \{a,b\}, X\}$ and $x_n = b$
x_n converges to b (expected) and x_n converges to c as well (unexpected)



Why? Roughly speakly, there are insufficient open subsets in X to distinguish the points b and c.

Exercise:

$(X, \mathcal{T})$ is said to be a Hausdorff space if for all $x, y \in X$ with $x_1 \neq x_2$ there exist $U, V \in \mathcal{T}$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$.
If $\{x_n\}$ is a convergent sequence in a Hausdorff space, then show the uniqueness of the limit.



8.2 Continuous Functions

Definition :

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $f: X \to Y$.

f is said to be continuous at $x \in X$ if

for all $V \in \mathcal{T}_Y$ containing $f(x)$, there exists $U \in \mathcal{T}_X$ containing x such that $f(U) \subseteq V$.

f is said to be continuous if it is continuous at every point in X.

Definition :

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $f: X \to Y$ be a bijection.

If both $f: X \to Y$ and $f^{-1}: Y \to X$ are continuous, then f is said to be a homeomorphism.

(i.e. X and Y have the same structure as topological space.)

Question :

Does it exist a homeomorphism between S^1 and $\mathbb{R}$? (Answer: No!)

Refer to Algebraic Topology.

8.3 Metric Spaces and Normed Spaces

Definition :

A metric on a set X is a function $d: X \times X \to \mathbb{R}$ such that

1) $d(x,y) \geq 0$ for all $x, y \in X$; $d(x,y) = 0$ if and only if $x = y$.
2) $d(x,y) = d(y,x)$ for all $x, y \in X$
3) $d(x,y) + d(y,z) \geq d(x,z)$ for all $x, y, z \in X$ (triangle inequality)

(Idea : $d: X \times X \to \mathbb{R}$ gives the "distance" between any two points in X.)

Then the set X with a metric $d: X \times X \to \mathbb{R}$ is said to be a metric space and it is denoted by (X, d)

Let $d: X \times X \to \mathbb{R}$ be a metric on X and let $x \in X$, $r > 0$,

Define $B_d(x, r) = \{ y \in X : d(x,y) < r \}$. It is called the ball centered at x with radius r.

Define $\mathcal{B}$ be the collection of all balls $B_d(x, r)$, where $x \in X$, $r > 0$.

Check $\mathcal{B}$ defines a basis on X and hence generates a topology on X which is called the metric topology.

Examples:

1) Define $d:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by $d(x,y)=|x-y|$

It is the standard or Euclidean metric on $\mathbb{R}$.

In general, if $\vec{x}=(x_1,x_2,\cdots,x_n)$, $\vec{y}=(y_1,y_2,\cdots,y_n)\in\mathbb{R}^n$

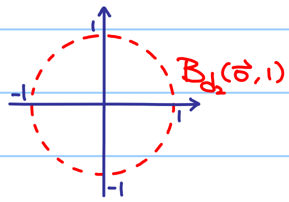
define $d:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ by $d(\vec{x},\vec{y})=\left[\sum_{i=1}^{n}(x_i-y_i)^2\right]^{\frac{1}{2}}$

It is the standard or Euclidean metric on $\mathbb{R}^n$.

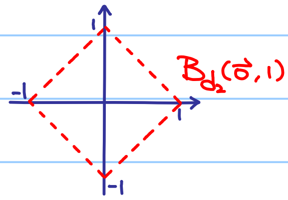
2) More generally, if $p\geq 1$, define $d_p:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ by $d(\vec{x},\vec{y})=\left[\sum_{i=1}^{n}|x_i-y_i|^p\right]^{\frac{1}{p}}$

It defines a metric on $\mathbb{R}^n$. (Check!)

$(\mathbb{R}^2, d_2)$

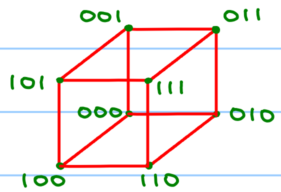


$(\mathbb{R}^2, d_1)$



3) Let $X=\{(a_1,a_2,\cdots,a_n): a_i\in\{0,1\}\}$

Define $d:X\times X\to\mathbb{R}$ by the number of components that are not the same.



distance between two points

= number of edges between them

Exercise:

Prove that (X,d) is a Hausdorff space.

Theorem: (Not surprising)

If (X,d) is a metric space equipped with the corresponding metric topology, $\{x_n\}\subseteq X$ is a sequence. $\{x_n\}$ converges to $c\in X$ if

for all $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that for all $n\geq N$, we have $d(x_n,c)<\varepsilon$.

Theorem: (Not surprising)

If (X,d_X) and (Y,d_Y) are metric spaces equipped with the corresponding metric topologies and $f:X\to Y$. f is continuous at $c\in X$ if

for all $\varepsilon>0$, there exists $\delta>0$ such that for all $x\in X$ with $d_X(x,c)<\delta$,

we have $d_Y(f(x),f(y))<\varepsilon$.

Definition:

Let X be a vector space over a field F (usually $F = \mathbb{R}$ or $\mathbb{C}$).

A norm is a function $\|\cdot\| : X \to \mathbb{R}$ such that

1) $\|\vec{x}\| \geq 0$ for all $\vec{x} \in X$; $\|\vec{x}\| = 0$ if and only if $\vec{x} = \vec{0}$.

2) $\|\alpha\vec{x}\| = |\alpha| \|\vec{x}\|$ for all $\vec{x} \in X$ and $\alpha \in F$, where $|\cdot|$ is the norm on F.

3) $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ for all $\vec{x}, \vec{y} \in X$ (Triangle inequality)

(Idea : $\|\cdot\| : X \to \mathbb{R}$ gives the "length" of a vector.)

Then the vector space X with norm $\|\cdot\| : X \to \mathbb{R}$ is said to be a normed space and it is denoted by $(X, \|\cdot\|)$.

Remark:

The vector space X is NOT necessary to be finite dimensional.

If X is a finite dimensional vector space over $\mathbb{R}$, then X is isomorphic to $\mathbb{R}^n$.

Examples:

1) Given $\vec{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, define $\|\vec{x}\| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$.

   It is called the standard or Euclidean norm on $\mathbb{R}^n$.

2) Let $\mathcal{C}([a,b])$ be the collection of continuous functions $f : [a,b] \to \mathbb{R}$.

   Check $\mathcal{C}([a,b])$ is a vector space.

   Let $p \geq 1$, define $\|\cdot\|_p : \mathcal{C}([a,b]) \to \mathbb{R}$ by $\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}$

   It is said to be L_p-norm on $\mathcal{C}([a,b])$.

Exercise:

If $(X, \|\cdot\|)$ is a normed space, define $d : X \times X \to \mathbb{R}$ by $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$.

Show that $d : X \times X \to \mathbb{R}$ defines a metric on X.

Refer to Functional Analysis.