§5 Continuous Functions

5.1 Continuous Functions

Recall: In secondary school, we have

"Definition": (Continuity)

A function $f: A \to \mathbb{R}$ is continuous at $c \in A$ if $\lim_{x \to c} f(x) = f(c)$.

Idea:

1) $\lim_{x \to c} f(x)$ exists;
2) $f(c)$ is well-defined;
3) They are the same.

Translate the above into δ-ε language:

Definition: (Continuity)

Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ and $c \in A$. We say that f is continuous at c if given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in A$ with $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in A \cap V_\delta(c))(f(x) \in V_\varepsilon(f(c)))$

If f fails to be continuous at c, then we say that f is discontinuous at c.

Theorem: (Sequential Criterion for Continuity)

A function $f: A \to \mathbb{R}$ is continuous at $c \in A$ if and only if for every sequence $\{x_n\} \subseteq A$ that converges to c, the sequence $\{f(x_n)\}$ converges to $f(c)$.

Exercises:

1) Prove the above theorem.
2) (a) Show that $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is continuous at every point.

   (b) Show that $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ is continuous at $x = 0$.
3) Show that $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ is discontinuous everywhere.

Remark :

In secondary school, we compute $\lim_{n\to\infty} \sqrt{4+\frac{1}{n}}$.

Let $x_n = 4+\frac{1}{n}$. Of course $\lim_{n\to\infty} 4+\frac{1}{n} = 4 = c$

However, why $\lim_{n\to\infty} \sqrt{x_n} \underset{??}{=} \sqrt{\lim_{n\to\infty} x_n} = \sqrt{4} = 2$?

Since we assume the continuity of $f(x) = \sqrt{x}$ at $x=4$,

by using the above theorem, $\lim_{n\to\infty} f(x_n) = f(c) = f(\lim_{n\to\infty} x_n)$

i.e. $\lim_{n\to\infty} \sqrt{4+\frac{1}{n}} = \sqrt{\lim_{n\to\infty} 4+\frac{1}{n}} = \sqrt{4} = 2$

Exercises :

1) Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by $f(x) = \begin{cases} 1 & \text{if } x=0 \\ 0 & \text{if } x\neq 0 \end{cases}$ and

$\{x_n\}$ be a sequence defined by $x_n = \frac{1}{n}$.

(a) Find $\lim_{n\to\infty} x_n$ and hence find $f(\lim_{n\to\infty} x_n)$.

(b) Find $f(x_n)$ and hence find $\lim_{n\to\infty} f(x_n)$.

(c) Are the results obtained in (a) and (b) the same ? Why ?

2) (a) Let $x>0$. Prove the existence of $\sqrt{x}$. (Hint : Recall the proof of existence of $\sqrt{2}$)

(b) Let $f: \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a function defined by $f(x) = \sqrt{x}$.

(i) Prove f is continuous at any $c>0$. (Hint : $|\sqrt{x}-\sqrt{c}| = \frac{1}{\sqrt{x}+\sqrt{c}} \cdot |x-c|$)

(ii) Prove f is continuous at 0. (Hint : if $0\leq x\leq 1$, then $\sqrt{x} > x$)

Definition :

A function $f: A\to\mathbb{R}$ is said to be a continuous function if

f is continuous at every point in A.

5.2 Combination of Continuous Functions

Theorem :

Let $A\subseteq\mathbb{R}$, $f,g: A\to\mathbb{R}$, $b\in\mathbb{R}$. Suppose f and g are continuous at $c\in A$.

(a) Then $f\pm g$, fg, bf are continuous at c.

(b) If $g(x)\neq 0$ for all $x\in A$, then f/g is continuous at c.

Direct consequences :

1) A polynomial function is continuous on $\mathbb{R}$.

2) A rational function is continuous at points that it is defined.

Theorem : (Composition of Continuous Functions)

Let $A, B \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ be functions such that $f(A) \subseteq B$.

If f is continuous at $c \in A$ and g is continuous at $f(c) \in B$,

then the composition $g \circ f : A \to \mathbb{R}$ is continuous at $c \in A$.

proof :

Let $\varepsilon > 0$, by continuity of g at $f(c)$, there exists $\delta'(\varepsilon) > 0$ such that

$|g(y) - g(f(c))| < \varepsilon$ for all $y \in B$ with $|y - f(c)| < \delta'(\varepsilon)$. —— (*)

By continuity of f at $c \in A$, there exists $\delta(\varepsilon) = \delta(\delta'(\varepsilon)) > 0$ such that

$|f(x) - f(c)| < \delta'$ for all $x \in A$ with $|x - c| < \delta(\varepsilon)$

Then, for all $x \in A$ with $|x - c| < \delta(\varepsilon)$,

$$|f(x) - f(c)| < \delta'(\varepsilon)$$

Apply (*) by letting $y = f(x)$

$$|g(f(x)) - g(f(c))| < \varepsilon$$

Immediate consequence.

Let $A, B \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ be continuous functions such that $f(A) \subseteq B$.

Then $g \circ f : A \to \mathbb{R}$ is a continuous function.

5.3 Continuous Functions on Intervals

Definition :

A function $f: A \to \mathbb{R}$ is said to be bounded on A if there exists a constant $M > 0$ such that $|f(x)| \le M$ for all $x \in A$.

Exercises :

1) Prove that $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x^2}$ is unbounded.

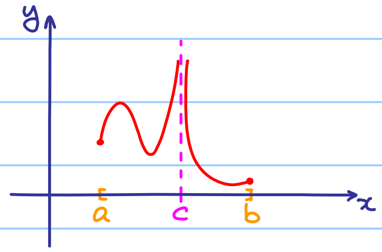
2) If $f: A \to \mathbb{R}$ is unbounded on A, prove that there exist a sequence $\{x_n\} \subseteq A$ such that $\lim_{n \to \infty} f(x_n) = +\infty$ or $-\infty$

Theorem: (Boundedness Theorem)

Let $I = [a,b]$ be a closed interval and let $f: I \to \mathbb{R}$ be continuous on I.

Then f is bounded on I.

Think:



unbounded

$\Rightarrow$ discontinuous at some point $c \in [a,b]$.

proof:

Suppose f is unbounded on I.

By the previous exercise (2), there exists $\{x_n\} \subseteq I$ such that $\lim\limits_{n\to\infty} f(x_n) = +\infty$ or $-\infty$.

By B-W theorem, since $\{x_n\} \subseteq I = [a,b]$, there exists a convergent subsequence $\{x_{n_r}\}$.

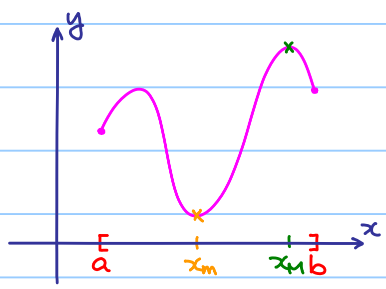
$\therefore$ We have $\lim\limits_{r\to\infty} x_{n_r} = l \in [a,b]$ but $\lim\limits_{r\to\infty} f(x_{n_r}) = +\infty$ or $-\infty$ (but NOT $f(l)$)

which contradicts to that f is continuous on I.

Definition:

Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$. We say that f has an **absolute maximum (minimum)** on A if there exists $x_M \in A$ ($x_m \in A$) such that $f(x_M) \geq f(x)$ ($f(x_m) \geq f(x)$) for all $x \in A$.

Think: If $f: [a,b] \to \mathbb{R}$ is continuous.



Continuity $\Rightarrow$ Existence of absolute maximum and minimum

Theorem:

Let $I = [a,b]$ and $f: I \to \mathbb{R}$ be a continuous function.

Then f has an absolute maximum and minimum on I.

proof :

By boundness theorem, $f(I)$ is bounded.

$\therefore \sup f(I)$ and $\inf f(I)$ exist.

Main issue : How to show they can be attained by some points in I ?

Let $s = \sup f(I)$.

For all $n \in \mathbb{N}$, $s - \frac{1}{n}$ is NOT an upper bound of $f(I)$,

there exists $x_n \in I$ such that $s - \frac{1}{n} < f(x_n) \le s$

$\therefore \lim_{n\to\infty} f(x_n) = s$

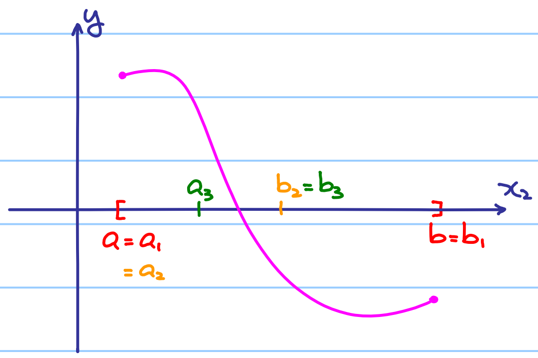
$\{x_n\}$ itself may NOT converge to a point in I, but B-W theorem helps !

By B-W theorem, there exists a convergent subsequence $\{x_{n_r}\}$ of $\{x_n\}$.

We define $x_M = \lim_{r\to\infty} x_{n_r} \in I$, then

$$
\begin{aligned}
f(x_M) &= f(\lim_{r\to\infty} x_{n_r}) \\
&= \lim_{r\to\infty} f(x_{n_r}) \\
&= s \quad (\ge f(x) \text{ for all } x \in I \text{ as } s = \sup f(I))
\end{aligned}
$$

Method of Bisection :



$f : [a,b] \longrightarrow \mathbb{R}$ is continuous

$f(a) > 0$ and $f(b) < 0$.

How to find / approximate a root ?

Bisect the interval, make sure $f(a_n)$ and $f(b_n)$ are in opposite signs each time

(if $f(a_n) = 0$ or $f(b_n) = 0$ for some $n \in \mathbb{N}$, it's done !).

Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c$ is a root of $f(x) = 0$.

Theorem : (Intermediate Value Theorem)

Let $f:[a,b] \longrightarrow \mathbb{R}$ be a continuous function.

If $f(a)f(b) < 0$ ($f(a)$ and $f(b)$ are in opposite signs),

then there exists $c \in (a,b)$ such that $f(c) = 0$.

proof:

WLOG, suppose that $f(a) > 0$ and $f(b) < 0$.

Construct $I_n = [a_n, b_n]$ as described above, then I_n is a nested sequence of closed intervals.

Also, $b_n - a_n = \frac{1}{2^{n-1}}(b-a)$. Therefore, $\inf\{b_n - a_n : n \in \mathbb{N}\} = 0$.

$\therefore$ There exists unique $c \in [a,b]$ such that $c \in [a_n, b_n]$ for all $n \in \mathbb{N}$.

Now, we claim $f(c) = 0$.

Note : By construction, $f(a_n) > 0$ and $f(b_n) < 0$ for all $n \in \mathbb{N}$.

$f(a_n) > 0$ for all $n \in \mathbb{N}$ $\Rightarrow$ $0 \leq \lim\limits_{n\to\infty} f(a_n) = f(\lim\limits_{n\to\infty} a_n) = f(c)$.

Similarly, $f(b_n) < 0$ for all $n \in \mathbb{N}$ $\Rightarrow$ $0 \geq \lim\limits_{n\to\infty} f(a_n) = f(\lim\limits_{n\to\infty} a_n) = f(c)$.

$\therefore f(c) = 0$.

Exercises :

1) Let $f(x) = x^2 - 2$.

Using the method of bisection to approximate the value of $\sqrt{2}$.

2) Prove Bolzano's Intermediate Value Theorem :

Let $f:[a,b] \longrightarrow \mathbb{R}$ be a continuous function.

If $f(a) < k < f(b)$ (or $f(a) > k > f(b)$)

then there exists $c \in (a,b)$ such that $f(c) = k$.

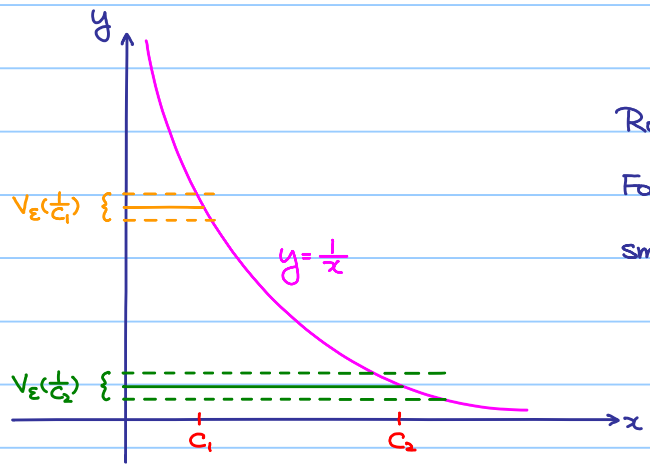
5.4 Uniform Continuity

Recall:

Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ and $c \in A$. We say that f is continuous at c if given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in A$ with $|x-c| < \delta$, then $|f(x)-f(c)| < \varepsilon$.

$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in A \cap V_\delta(c))(f(x) \in V_\varepsilon(f(c)))$

But, in fact, the choice of δ depends on $c \in A$ as well.



Rough idea:

For the same $\varepsilon > 0$, we have to choose a smaller δ at c_1 comparing to δ at c_2.

By the above, we have

$f: A \to \mathbb{R}$ is a continuous function if

$(\forall c \in A, \varepsilon > 0)(\exists \delta(c,\varepsilon) > 0)(\forall x \in A \text{ and } |x-c| \leq \delta(c,\varepsilon))(|f(x)-f(c)| < \varepsilon)$

Question:

Can we choose δ uniformly, i.e. δ depends on ε only (but NOT $c \in A$ any more)?

Definition:

Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$. We say that f is uniformly continuous on A if given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, c \in A$ with $|x-c| < \delta$, then $|f(x)-f(c)| < \varepsilon$.

$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, c \in A \text{ and } |x-c| < \delta)(|f(x)-f(c)| < \varepsilon)$

Natural consequence:

Uniformly continuous on A $\Rightarrow$ continuous on A

But the converse is NOT true.

Exercise :

Write down the negation of the above definition.

Ans : f is NOT uniformly continuous on A if

$(\exists \varepsilon>0)(\forall \delta>0)(\exists x, c\in A \text{ and } |x-c|<\delta)(|f(x)-f(c)|\geq\varepsilon)$

Rough idea : No matter how small δ is, we can always find two points x and c such that $|x-c|<\delta$ (i.e. x and c are close)

but $|f(x)-f(c)|>\varepsilon$ (i.e. $f(x)$ and $f(c)$ are NOT close).

Theorem :

Let $A\subseteq\mathbb{R}$ and $f:A\to\mathbb{R}$. f is NOT uniformly continuous on A if and only if there exists $\varepsilon_0>0$ and two sequences $\{x_n\}$ and $\{u_n\}$ in A such that $\lim_{n\to\infty} x_n-c_n=0$ but $|f(x_n)-f(c_n)|\geq\varepsilon_0$ for all $n\in\mathbb{N}$.

proof :

$(\Rightarrow)$: $(\exists\varepsilon_0>0)(\forall\delta>0)(\exists x, c\in A \text{ and } |x-c|<\delta)(|f(x)-f(c)|\geq\varepsilon_0)$

In particular, take $\delta=\frac{1}{n}$, $n\in\mathbb{N}$

there exists $x_n, c_n\in A$ such that $|x_n-c_n|<\delta=\frac{1}{n}$ and $|f(x_n)-f(c_n)|\geq\varepsilon_0$

Since $|x_n-c_n|<\frac{1}{n}$ for all $n\in\mathbb{N}$, $\lim_{n\to\infty}x_n-c_n=0$

$(\Leftarrow)$: By assumption, there exists $\varepsilon_0>0$ and two sequences $\{x_n\}$ and $\{u_n\}$ in A such that $\lim_{n\to\infty}x_n-c_n=0$ but $|f(x_n)-f(c_n)|\geq\varepsilon_0$ for all $n\in\mathbb{N}$.

For that $\varepsilon_0>0$, let $\delta>0$, there exists $N\in\mathbb{N}$ such that $\frac{1}{N}<\delta$

Then, $|x_N-c_N|<\frac{1}{N}<\delta$ and $|f(x_N)-f(c_N)|\geq\varepsilon_0$.

Exercises :

1) If $f:(0,1)\to\mathbb{R}$ is defined by $f(x)=\frac{1}{x}$. Show that f is NOT uniformly continuous on $(0,1)$.

(Hint: consider $x_n=\frac{1}{n}$ and $c_n=\frac{1}{n+1}$)

2) If $f:\mathbb{R}^+\to\mathbb{R}$ is defined by $f(x)=x^2$. Show that f is NOT uniformly continuous on $\mathbb{R}^+$.

(Hint: consider $x_n=n+\frac{1}{n}$ and $c_n=n$)

(Note: $\lim_{n\to\infty}x_n$ and $\lim_{n\to\infty}c_n$ do NOT exist but $\lim_{n\to\infty}x_n-c_n=0$.)

Uniformly continuity $\underset{\not\Leftarrow}{\Rightarrow}$ Continuity

However, the converse statement is true for some cases, such as :

Theorem :

Let I be a closed bounded interval and let $f: I \to \mathbb{R}$ be a function.

If f is continuous, then f is uniformly continuous.

proof :

Suppose the contrary, there exists $\varepsilon_0 > 0$ and two sequences $\{x_n\}$ and $\{u_n\}$ in A such that

$\lim_{n\to\infty} x_n - c_n = 0$ but $|f(x_n) - f(c_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$.

B-W Theorem $\Rightarrow$ there exists a convergent subsequence $\{x_{n_r}\} \subseteq \{x_n\}$

Let $\lim_{r\to\infty} x_{n_r} = \ell$, by continuity of f, $\lim_{r\to\infty} f(x_{n_r}) = f(\ell)$.

Exercise : Show $\lim_{r\to\infty} c_{n_r} = \ell$.

(Hint : $|c_{n_r} - \ell| \leq |c_{n_r} - x_{n_r}| + |x_{n_r} - \ell|$.)

$\therefore \lim_{r\to\infty} f(c_{n_r}) = \ell$ and so $\lim_{r\to\infty} f(x_{n_r}) - f(c_{n_r}) = f(\ell) - f(\ell) = 0$ (Contradiction !)

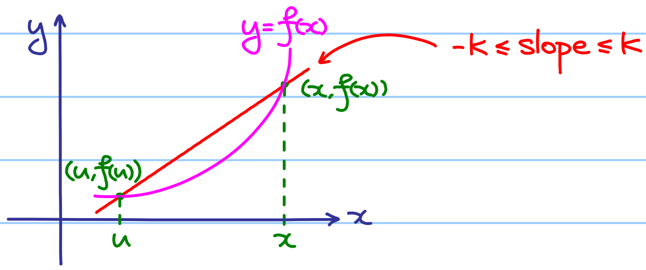
Definition : (Lipschitz Function)

Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$. If there exists a constant $K > 0$ such that

$|f(x) - f(u)| \leq K|x-u|$ for all $x, u \in A$, then f is said to be a Lipschitz function on A.

Actually, if $x = u$, $|f(x) - f(u)| = |x-u| = 0$. (Nothing Interesting !)

If $x \neq u$, Lipschitz condition means $\left|\frac{f(x)-f(u)}{x-u}\right| \leq K$ for all $x, u \in A$,



Note : A continuous function is NOT necessary a Lipschitz function !

Exercise :

Let $f: [0,1] \to \mathbb{R}$ be a function defined by $f(x) = \sqrt{x}$. Show that f is NOT a Lipschitz function.

Continuity $\not\Rightarrow$ Lipschitz

but how about $\Leftarrow$?

In fact, we prove a even stronger result.

Theorem:

If $f: A \to \mathbb{R}$ is a Lipschitz function on A, then f is uniformly continuous on A.

proof:

Given $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{K} > 0$, then for all $x, u \in A$ with $|x-u| < \delta = \frac{\varepsilon}{K} > 0$,

we have $|f(x) - f(u)| \leq K|x-u| < K \cdot \frac{\varepsilon}{K} = \varepsilon$.

Conclusion:

Lipschitz $\overset{①}{\Rightarrow}$ Uniformly Continuity $\Rightarrow$ Continuity

"$\overset{②}{\Leftarrow}$" is true when the domain is a closed bounded interval

Example:

If $f: [0, +\infty)$ is a function defined by $f(x) = \sqrt{x}$. Show that $f(x)$ is uniformly continuous on $[0, +\infty)$.

Think: Troubles (i) $[0, +\infty)$ is NOT bounded. (cannot use ②)

(ii) f is NOT Lipschitz on $[0, +\infty)$, even NOT on $[0, 1]$. (cannot use ①)

Idea: Combine them together.

(i) ② $\Rightarrow$ f is uniformly continuous on $[0, 2]$.

(ii) ① $\Rightarrow$ f is uniformly continuous on $[1, +\infty)$.

Result:

Let $\varepsilon > 0$

(i) there exists $\delta_1 > 0$ such that for all $x, c \in [0, 2]$ with $|x-c| < \delta_1$, we have $|f(x) - f(c)| < \varepsilon$

(ii) there exists $\delta_2 > 0$ such that for all $x, c \in [1, +\infty)$ with $|x-c| < \delta_2$, we have $|f(x) - f(c)| < \varepsilon$

Take $\delta = \min\{\delta_1, \delta_2, 1\} < 0$ why? (*)

If $x, u \in [0, +\infty)$ with $|x-u| < \delta$

(*) $\Rightarrow$ both x, u lie on $[0, 2]$ or both x, u lie on $[1, +\infty)$

then either (i) and (ii) $\Rightarrow$ $|f(x) - f(u)| < \varepsilon$.