MAT6082 Topics in Analysis

2^{nd} term, 2015-16

Teacher: Professor Ka-Sing Lau

Schedule: Wednesday, 2.30-5.00 pm

Venue: LSB 222

Topics: Introduction to Stochastic Calculus

In the past thirty years, there has been an increasing demand of stochastic calculus in mathematics as well as various disciplines such as mathematical finance, pde, physics and biology. The course is a rigorous introduction to this topic. The material include conditional expectation, Markov property, martingales, stochastic processes, Brownian motions, Ito's calculus, and stochastic differential equations.

Prerequisites

Students are expected to have good background in real analysis, probability theory and some basic knowledge of stochastic processes.

References:

1. A Course in Probability Theory, K.L. Chung, (1974).
2. Measure and Probability, P. Billingsley, (1986).
3. Introduction to Stochastic Integration, H.H. Kuo, (2006).
4. Intro. to Stochastic Calculus with Application, F. Klebaner, (2001).
5. Brownian Motion and Stoch. Cal., I. Karatzas and S. Shreve, (1998).
6. Stoch. Cal. for Finance II– Continuous time model, S. Shreve, (2004).

Everyone knows calculus deals with deterministic objects. On the other hand *stochastic calculus* deals with random phenomena. The theory was introduced by Kiyosi Ito in the 40's, and therefore stochastic calculus is also called *Ito calculus*. Besides its interest in mathematics, it has been used extensively in statistical mechanics in physics, the filter and control theory in engineering. Nowadays it is very popular in the option price and hedging in finance. For example the well-known Black-Scholes model is

$$dS(t) = rS(t)dt \ + \ \sigma S(t)dB(t)$$

where $S(t)$ is the stock price, σ is the volatility, and r is the interest rate, and $B(t)$ is the Brownian motion. The most important notion for us is the Brownian motion. As is known the botanist R. Brown (1828) discovered certain zigzag random movement of pollens suspended in liquid. A. Einstein (1915) argued that the movement is due to bombardment of particle by the molecules of the fluid. He set up some basic equations of Brownian motion and use them to study diffusion. It was N. Wiener (1923) who made a rigorous study of the Brownian motion using the then new theory of Lebesgue measure. Because of that a Brownian motion is also frequently called a Wiener process.

Just like calculus is based on the *fundamental theorem of calculus*, the Ito calculus is based on the *Ito Formula*: Let f be a twice differentiable function on $\mathbb{R}$, then

$$f(B(t)) - f(B(0)) = \int_0^T f'(B(t))dB(t) + \frac{1}{2}\int_0^T f''(B(t))dt$$

where $B(0) = 0$ to denote the motion starts at 0. There are formula for integration, for example, we have

$$\int_0^T B(t)dB(t) = \frac{1}{2}B(t)^2 - \frac{1}{2}T; \qquad \int_0^T tdB(t) = TB(T) - \int_0^T B(t)dt.$$

In this course, the prerequisite is real analysis and basic probability theory. In real analysis, one needs to know σ-fields, measurable functions, measures and integration theory, various convergence theorems, Fubini theorem and the Radon-Nikodym theorem. We will go through some of the probability theory on conditional expectation, optional r.v. (stopping time), Markov property, martingales ([1], [2]). Then we will go onto study the Brownian motion ([2], [3], [5]), the stochastic integration and the Ito calculus ([3], [4], [5]).

Chapter 1

Basic Probability Theory

1.1 Preliminaries

Let Ω be a set and let $\mathcal{F}$ be a family of subsets of Ω, $\mathcal{F}$ is called a *field* if it satisfies

(i) $\emptyset,\ \Omega \in \mathcal{F}$;

(ii) for any $A \in \mathcal{F},\ A^c \in \mathcal{F}$;

(iii) for any $A,\ B \in \mathcal{F},\ A \cup B \in \mathcal{F}$ (hence $A \cap B \in \mathcal{F}$).

It is called a σ*-field* if (iii) is replaced by

(iii)$'$ for any $\{A_n\}_{n=1}^{\infty} \subset \mathcal{F},\ \cup_{n=1}^{\infty} A_n \in \mathcal{F}$ (hence $\cap_{n=1}^{\infty} A_n \in \mathcal{F}$).

If $\Omega = \mathbb{R}$ and $\mathcal{F}$ is the smallest σ-field generated by the open sets, then we call it the Borel field and denote by $\mathcal{B}$.

A *probability space* is a triple $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F}$ is a σ-field in Ω, and $P : \mathcal{F} \to [0,1]$ satisfies

(i) $P(\Omega) = 1$

(ii) countable additivity : if $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ is a disjoint family, then

$$P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n).$$

We call Ω a sample space, $A \in \mathcal{F}$ an event (or measurable set) and P a probability measure on Ω; an element $\omega \in \Omega$ is called an outcome.

Theorem 1.1.1. *(Caratheodory Extension Theorem) Let $\mathcal{F}_0$ be a field of subsets in Ω and let $\mathcal{F}$ be the $\sigma-$ field generated by $\mathcal{F}_0$. Let $P: \ \mathcal{F}_0 \to [0,1]$ satisfies (i) and (ii) (on $\mathcal{F}_0$). Then P can be extended uniquely to $\mathcal{F}$, and $(\Omega, \mathcal{F}, P)$ is a probability space.*

The proof of the theorem is to use the outer measure argument.

Example 1. Let $\Omega = [0,1]$, let $\mathcal{F}_0$ be the family of set consisting of finite disjoint unions of half open intervals $(a,b]$ and $[0,b]$, Let $P([a,b)) = |b-a|$. Then $\mathcal{F}$ is the Borel field and P is the Lebesgue measure on $[0,1]$.

Example 2. Let $\{(\Omega_n, \mathcal{F}_n, P_n)\}_n$ be a sequence of probability spaces. Let $\Omega = \prod_{n=1}^{\infty} \Omega_n$ be the product space and let $\mathcal{F}_0$ be the family of subsets of the form $E = \prod_{n=1}^{\infty} E_n$, where $E_n \in \mathcal{F}_n$, $E_n = \Omega_n$ except for finitely many n. Define

$$P(E) = \prod_{n=1}^{\infty} P(E_n)$$

Let $\mathcal{F}$ be the σ-field generated $\mathcal{F}_0$, then $(\Omega, \mathcal{F}, P)$ is the standard infinite product measure space.

Example 3. (Kolmogorov Extension Theorem) Let P_n be probability measures on $(\prod_{k=1}^{n} \Omega_k, \mathcal{F}_n)$ satisfying the following consistency condition: for $m \leq n$

$$P_n \circ {\pi_{nm}}^{-1} = P_m$$

where $\pi_{nm}(x_1 \cdots x_n) = (x_1 \cdots x_m)$. On $\Omega = \prod_{k=1}^{\infty} \Omega_k$, we let $\mathcal{F}_0$ be the field of sets $F = E \times \prod_{k=n+1}^{\infty} \Omega_k$, $E \in \mathcal{F}_n$ and let

$$P(F) = P_n(E).$$

Then this defines a probability spaces $(\Omega,\ \mathcal{F}, P)$, where $\mathcal{F}$ is the σ-field generated by $\mathcal{F}_0$.

Remark: The probability space in Example 2 is the underlying space for a sequence of independent random variables. Example 3 is for more general sequence of random variables (with the consistency condition).

A random variable (r.v.) X on $(\Omega, \mathcal{F})$ is an (extended) real valued function $X : (\Omega, \mathcal{F}) \to \mathbb{R}$ such that for any Borel subset B of $\mathbb{R}$,

$$X^{-1}(B) = \{\omega : X(\omega) \in B\} \in \mathcal{F}.$$

(i.e. X is $\mathcal{F}$-measurable). We denote this by $X \in \mathcal{F}$. It is well known that

– For $X \in \mathcal{F}$, X is either a simple function (i.e., $\sum_{k=1}^{n} a_k \chi_{A_k}(\omega)$ where $A_k \in \mathcal{F}$), or is the pointwise limit of a sequence of simple functions.

– Let $X \in \mathcal{F}$ and g is a Borel measurable function, then $g(X) \in \mathcal{F}$.

– If $\{X_n\} \subseteq \mathcal{F}$ and $\lim_{n\to\infty} X_n = X$, then $X \in \mathcal{F}$.

– Let $\mathcal{F}_X$ be the σ-field generated by X, i.e., the sub-σ-field $\{X^{-1}(B) : B \in \mathcal{B}\}$. Then for any $Y \in \mathcal{F}_X$, $Y = \varphi(X)$ for some extended-valued Borel function φ on $\mathbb{R}$.

Sketch of proof ([1, p.299]): First prove this for simple r.v. Y so that $Y = \phi(X)$ for some simple function ϕ. For a bounded r.v. $Y \geq 0$, we can find a sequence of increasing simple functions $\{Y_n\}$ such that $Y_n = \phi_n(X)$ and

$Y_n \nearrow Y$. Let $\phi(x) = \overline{\lim}_n \phi_n(x)$, hence $Y = \phi(X)$. Then prove Y for the general case.

A r.v. $X : (\Omega, \mathcal{F}) \to \mathbb{R}$ induces a *distribution (function)* on $\mathbb{R}$:

$$F(x) = F_X(x) = P(X \le x).$$

It is a non-decreasing, right continuous function with $\lim_{n\to-\infty} F(x) = 0$, $\lim_{n\to\infty} F(x) = 1$. The distribution defines a measure μ

$$\mu((a,b]) = F(b) - F(a)$$

(use the Caratheodory Extension Theorem here). More directly, we can define μ by

$$\mu(B) = P(X^{-1}(B)) \;, \quad B \in \mathcal{B}.$$

The jump of F at x is $F(x) - F(x-) = P(X = x)$. A r.v. X is called a *discrete* if F is a jump function; X is called a *continuous* r.v. if F is continuous, i.e., $P(X = x) = 0$ for each $x \in \mathbb{R}$, and X is said to have a density function $f(x)$ if F is absolutely continuous with the Lebesgue measure and $f(x) = F'(x)$ a.e., equivalently $F(x) = \int_{-\infty}^{x} f(y)dy$.

For two random variables X, Y on $(\Omega, \mathcal{F})$, the random vector (X, Y) : $(\Omega, \mathcal{F}) \to \mathbb{R}^2$ induces a distribution F on $\mathbb{R}^2$

$$F(x, y) = P(X \le x, \; Y \le y)$$

and F is called the joint distribution of (X, Y), the corresponding measure μ is given by

$$\mu((a,b] \times (c,d]) = F(b,d) - F(a,d) - F(b,c) + F(a,c),$$

Similarly we can define the joint distribution $F(x_1 \cdots x_n)$ and the corresponding measure.

For a sequence of r.v., $\{X_n\}_{n=1}^{\infty}$, there are various notions of convergence.

(a) $X_n \to X$ a.e. (or a.s.) if $\lim_{n\to\infty} X_n(\omega) = X(\omega)$ (pointwise) for $\omega \in \Omega \backslash E$ where $P(E) = 0$.

(b) $X_n \to X$ in probability if for any $\epsilon > 0$, $\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = 0$.

(c) $X_n \to X$ in distribution if $F_n(x) \to F(x)$ at every point x of continuity. It is equivalent to $\mu_n \to \mu$ vaguely i.e., $\mu_n(f) \to \mu(f)$ for all $f \in C_0(\mathbb{R})$, the space of continuous functions vanish at ∞ (detail in [1]).

The following relationships are basic ([1] or Royden): $(a) \Rightarrow (b) \Rightarrow (c)$; $(b) \Rightarrow (a)$ on some subsequence. On the other hand we cannot expect (c) to imply (b) as the distribution does not determine X. For example consider the interval $[0,1]$ with the Lebesgue measure, the r.v.'s $X_1 = \chi_{[0,\frac{1}{2}]}$, $X_2 = \chi_{[\frac{1}{2},1]}$, $X_3 = \chi_{[0,\frac{1}{4}]} + \chi_{[\frac{3}{4},1]}$ all have the same distribution.

The expectation of a random variable is defined as

$$E(X) = \int_{\Omega} X(\omega) dP(\omega) = \int_{-\infty}^{\infty} x dF(x) \ (= \int_{-\infty}^{\infty} x d\mu(x))$$

and for a Borel measurable h, we have

$$E(h(X)) = \int_{\Omega} h(X(\omega)) dP(\omega) = \int_{-\infty}^{\infty} h(x) dF(x).$$

The most basic convergence theorems are:

(a) Fatou lemma:

$$X_n \geq 0, \quad \text{then} \ \ E(\underline{\lim}_{n\to\infty} X_n) \ \leq \ \underline{\lim}_{n\to\infty} E(X_n).$$

(b) Monotone convergence theorem:

$$X_n \geq 0, \ X_n \nearrow X, \quad \text{then} \quad \lim_{n\to\infty} E(X_n) = E(X).$$

(c) Dominated convergence theorem:

$$|X_n| \le Y,\ E(Y) < \infty \text{ and } X_n \to X\ a.e., \text{ then } \lim_{n\to\infty} E(X_n) = E(X).$$

We say that $X_n \to X$ in $L^p, p > 0$ if $E(|X|^p) < \infty$ and $E(|X_n - X|^p) \to 0$ as $n \to \infty$. It is known that L^p convergence implies convergence in probability. The converse also holds if we assume further $E(|X_n|^p) \to E(|X|^p) < \infty$ ([1], p.97).

Two events $A, B \in \mathcal{F}$ are said to be *independent* if

$$P(A \cap B) = P(A)P(B).$$

Similarly we say that the events $A_1, \cdots A_n \in \mathcal{F}$ are independent if for any subsets $A_{j_1}, \cdots, A_{j_k}$,

$$P(\bigcap_{i=1}^{k} A_{j_i}) = \prod_{i=1}^{k} P(A_{j_i}).$$

Two sub-σ-fields $\mathcal{F}_1$ and $\mathcal{F}_2$ are said to be independent if any choice of sets of each of these σ-fields are independent. Two r.v.'s X, Y are independent if the σ-fields $\mathcal{F}_X$ and $\mathcal{F}_Y$ they generated are independent. Equivalently we have

$$P(X \le x,\ Y \le y) = P(X \le x)\ P(Y \le y),$$

(i.e., the joint distribution equals the product of their marginal distributions). We say that $X_1 \cdots X_n$ are independent if for any $X_{i_1} \cdots X_{i_k}$, their joint distribution is a product of their marginal distributions.

Proposition 1.1.2. *Let X, Y be independent, then $f(X)$ and $g(Y)$ are independent for any Borel measurable functions f and g.*

Exercises

1. Can you identify the interval $[0,1]$ with the Lebesgue measure to the probability space for tossing a fair coin repeatedly?

2. Prove Proposition 1.1.2.

3. Suppose that $\sup_n |X_n| \leq Y$ and $E(Y) < \infty$. Show that

$$E(\overline{\lim}_{n\to\infty} X_n) \;\geq\; \overline{\lim}_{n\to\infty} E(X_n)$$

4. If $p > 0$ and $E(|X|^p) < \infty$, then $\lim_{n\to\infty} x^p P(|X| > x) = 0$. Conversely, if $\lim_{n\to\infty} x^p P(|X| > x) = 0$, then $E(|X|^{p-\epsilon}) < \infty$ for $0 < \epsilon < p$.

5. For any d.f. F and any $a \geq 0$, we have

$$\int_{-\infty}^{\infty} (F(x+a) - F(x))dx = a$$

6. Let X be a positive r.v. with a distribution F, then

$$\int_0^{\infty} (1 - F(x))\; dx = \int_0^{\infty} x\; dF(x).$$

and

$$E(X) = \int_0^{\infty} P(X > x)\; dx = \int_0^{\infty} P(X \geq x)\; dx$$

7. Let $\{X_n\}$ be a sequence of identically distributed r.v. with finite mean, then

$$\lim_n \frac{1}{n} E(\max_{1\leq j\leq n} |X_j|) = 0.$$

(Hint: use Ex.6 to express the mean of the maximum)

8. If X_1, X_2 are independent r.v.'s each takes values $+1$ and -1 with probability $\frac{1}{2}$, then the three r.v.'s $\{X_1, X_2, X_1X_2\}$ are pairwise independent but not independent.

9. A r.v. is independent of itself if and only if it is constant with probability one. Can X and $f(X)$ be independent when $f \in \mathcal{B}$?

10 . Let $\{X_j\}_{j=1}^n$ be independent with distributions $\{F_j\}_{j=1}^n$. Find the distribution for $\max_j X_j$ and $\min_j X_j$.

11. If X and Y are independent and $E(|X+Y|^p) < \infty$ for some $p > 0$, then $E(|X|^p) < \infty$ and $E(|Y|^p) < \infty$.

12. If X and Y are independent, $E(|X|^p) < \infty$ for some $p \geq 1$, and $E(Y) = 0$, then $E(|X+Y|^p) \geq E(|X|^p)$.

1.2 Conditional Expectation

Let $\Lambda \in \mathcal{F}$ with $P(\Lambda) > 0$, we define

$$P(E|\Lambda) = \frac{P(\Lambda \cap E)}{P(\Lambda)} \qquad \text{where} \quad P(\Lambda) > 0.$$

It follow that for a discrete random vector (X, Y),

$$P(Y = y|X = x) = \begin{cases} \dfrac{P(Y = y,\ X = x)}{P(X = x)} \ , & \text{if} \ \ P(X = x) > 0 \ , \\ 0 \ , & \text{otherwise.} \end{cases}$$

Moreover if (X, Y) is a continuous random variable with joint density $f(x, y)$, the conditional density of Y given $X = x$ is

$$f(y|x) = \begin{cases} \dfrac{f(x,y)}{f_X(x)} \ , & \text{if} \ \ f_X(x) > 0 \ , \\ 0 \ , & \text{otherwise} \ . \end{cases}$$

where $f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy$ is the marginal density. The conditional expectation of Y given $X = x$ is

$$E(Y|X = x) = \int_{-\infty}^{\infty} yf(y|x)dy.$$

Note that

$$g(x) := E(Y|X = x) \quad \text{is a function on} \ \ x \ ,$$

and hence

$$g(X(\cdot)) := E(Y|X(\cdot)) \quad \text{is a r.v. on} \ \ \Omega \ . \tag{1.2.1}$$

In the following we have a more general consideration for the conditional expectation (and also the conditional probability): $E(Y|\mathcal{G})$ where $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$.

First let us look at a special case where $\mathcal{G}$ is generated by a measurable partition $\{\Lambda_n\}_n$ of Ω (each member in $\mathcal{G}$ is a union of $\{\Lambda_n\}_n$). Let Y be an integrable r.v., then

$$E(Y|\Lambda_n) = \int_\Omega Y(\omega)dP_{\Lambda_n}(\omega) = \frac{1}{P(\Lambda_n)}\int_{\Lambda_n} Y(\omega)dP(\omega). \tag{1.2.2}$$

(Here $P_{\Lambda_n}(\cdot) = \frac{P(\,\cdot\,\cap\Lambda_n)}{P(\Lambda_n)}$ is a probability measure for $P(\Lambda_n) > 0$). Consider the random variable (as in (1.2.1))

$$Z(\cdot) = E(Y|\mathcal{G})(\cdot) := \sum_n E(Y|\Lambda_n)\chi_{\Lambda_n}(\cdot) \in \mathcal{G}.$$

It is easy to see that if $\omega \in \Lambda_n$, then $Z(\omega) = E(Y|\Lambda_n)$, and moreover

$$\int_\Omega E(Y|\mathcal{G})dP = \sum_n \int_{\Lambda_n} E(Y|\mathcal{G})dP = \sum_n E(Y|\Lambda_n)P(\Lambda_n) = \int_\Omega YdP\ .$$

We can also replace Ω by $\Lambda \in \mathcal{G}$ and obtain

$$\int_\Lambda E(Y|\mathcal{G})dP = \int_\Lambda YdP \qquad \forall\ \Lambda \in \mathcal{G}.$$

Recall that for μ, ν two σ-finite measures on $(\Omega, \mathcal{F})$ and $\mu \geq 0$, ν is called *absolutely continuous* with respect to μ $(\nu \ll \mu)$ if for any $\Lambda \in \mathcal{F}$ and $\mu(\Lambda) = 0$, then $\nu(\Lambda) = 0$. The Radon-Nikodym theorem says that there exists $g = \dfrac{d\nu}{d\mu}$ such that

$$\nu(\Lambda) = \int_\Lambda g d\mu \qquad \forall\ \ \Lambda \in \mathcal{F}.$$

Theorem 1.2.1. *If $E(|Y|) < \infty$ and $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$, t hen there exists a unique $\mathcal{G}$-measurable r.v., denote by $E(Y|\mathcal{G}) \in \mathcal{G}$, such that*

$$\int_\Lambda YdP = \int_\Lambda E(Y|\mathcal{G})\ dP \qquad \forall\ \Lambda \in \mathcal{G}.$$

Proof. Consider the set-valued function

$$\nu(\Lambda) = \int_\Lambda YdP \qquad \Lambda \in \mathcal{G}.$$

Then ν is a "signed measure" on $\mathcal{G}$. It satisfies

$$P(\Lambda) = 0 \implies \nu(\Lambda) = 0.$$

Hence ν is absolutely continuous with respect to P. By the Radon-Nikodym theorem, the derivative $g = \frac{d\nu}{dP} \in \mathcal{G}$ and

$$\int_\Lambda Y dP = v(\Lambda) = \int_\Lambda g dP \qquad \forall \ \Lambda \in \mathcal{G}.$$

This g is unique: for if we have $g_1 \in \mathcal{G}$ satisfies the same identity,

$$\int_\Lambda Y dP = v(\Lambda) = \int_\Lambda g_1 dP \qquad \forall \ \Lambda \in \mathcal{G}.$$

Let $\Lambda = \{g > g_1\} \in \mathcal{G}$, then $\int_\Lambda (g - g_1) dP = 0$ implies that $P(\Lambda) = 0$. We can reverse g and g_1 and hence we have $P(g \neq g_1) = 0$. It follows that $g = g_1$ $\mathcal{G}$-a.e.

Definition 1.2.2. *Given an integrable r.v. Y and a sub-σ-field $\mathcal{G}$, we say that $E(Y|\mathcal{G})$ is the* conditional expectation *of Y with respect to $\mathcal{G}$ (also denote by $E_\mathcal{G}(Y)$) if it satisfies*

(a) $E(Y|\mathcal{G}) \in \mathcal{G}$;

(b) $\int_\Lambda Y dP = \int_\Lambda E(Y|\mathcal{G}) dP \quad \forall \ \Lambda \in \mathcal{G}$.

If $Y = \chi_\Delta \in \mathcal{F}$, we define $P(\Delta|\mathcal{G}) = E(\chi_\Delta|\mathcal{G})$ and call this the conditional probability with respect to $\mathcal{G}$.

Note that the *conditional probability* can be put in the following way:

(a)' $\quad P(\Delta|\mathcal{G}) \in \mathcal{G}$;

(b)' $\quad P(\Delta \cap \Lambda) = \int_\Lambda P(\Delta|\mathcal{G}) dP \quad \forall \ \Lambda \in \mathcal{G}$.

It is a simple exercise to show that the original definition of $P(\Delta|\Lambda)$ agrees with this new definition by taking $\mathcal{G} = \{\emptyset, \ \Lambda, \ \Lambda^c, \ \Omega\}$.

Note that $E(Y|\mathcal{G})$ is "almost everywhere" defined, and we call one such function as a "version" of the conditional expectation. For brevity we will not mention the "a.e." in the conditional expectation unless necessary. If $\mathcal{G}$ is the sub-σ-field $\mathcal{F}_X$ generated by a r.v. X, we write $E(Y|X)$ instead of $E(Y|\mathcal{F}_X)$. Similarly we can define $E(Y|X_1, \cdots, X_n)$.

Proposition 1.2.3. *For $E(Y|X) \in \mathcal{F}_X$, there exists an extended-valued Borel measurable φ such that $E(Y|X) = \varphi(X)$, and φ is given by*

$$\varphi = \frac{d\lambda}{d\mu} ,$$

where $\lambda(B) = \int_{X^{-1}(B)} Y dP, \quad B \in \mathscr{B}$, and μ is the associated probability of the r.v. X on $\mathbb{R}$.

Proof. Since $E(Y|X) \in \mathcal{F}_X$, we can write $E(Y|X) = \varphi(X)$ for some Borel measurable φ (see §1). For $\Lambda \in \mathcal{F}$, there exists $B \in \mathcal{B}$ such that $\Lambda = X^{-1}(B)$. Hence

$$\int_\Lambda E(Y|X)dP = \int_\Omega \chi_B(X)\varphi(X)dP = \int_{\mathbb{R}} \chi_B(X)\varphi(X)d\mu = \int_B \varphi(x)d\mu$$

On the other hand by the definition of conditional probability,

$$\int_\Lambda E(Y|X)dP = \int_{X^{-1}(B)} Y dP = \lambda(B).$$

It follows that $\lambda(B) = \int_B \varphi(x)d\mu$ for all $B \in \mathcal{B}$. Hence $\varphi = \dfrac{d\lambda}{d\mu}$. $\square$

The following are some simple facts of the conditional expectation:

– If $\mathcal{G} = \{\phi, \Omega\}$, then $E(Y|\mathcal{G})$ is a constant function and equals $E(Y)$.

– If $\mathcal{G} = \{\phi, \Lambda, \Lambda^c, \Omega\}$, then $E(Y|\mathcal{G})$ is a simple function which equals $E(Y|\Lambda)$ on Λ, and equals $E(Y|\Lambda^c)$ on Λ^c,

– If $\mathcal{G} = \mathcal{F}$ or $Y \in \mathcal{G}$, then $E(Y|\mathcal{G}) = Y$.

– If (X, Y) has a joint density function, then $E(Y|X)$ coincides with the expression in (1.2.1).

Using the defining relationship of conditional expectation, we can show that the linearity, the basic inequalities and the convergence theorems for $E(\cdot)$ also hold for $E(\cdot\ |\mathcal{G})$. For example we have

Proposition 1.2.4. *(Jensen inequality) If φ is a convex function on $\mathbb{R}$, and Y and $\varphi(Y)$ are integrable r.v., then for each sub-σ-algebra $\mathcal{G}$,*

$$\varphi\big(E(Y|\mathcal{G})\big) \le E\big(\varphi(Y)|\mathcal{G}\big)$$

Proof. If Y is a simple r.v., then $Y = \sum_{j=1}^n y_j \chi_{\Lambda_j}$ with $\Lambda \in \mathcal{F}$. It follows that

$$E(Y|\mathcal{G}) = \sum_{j=1}^n y_j E(\chi_{\Lambda_j}|\mathcal{G}) = \sum_{j=1}^n y_j P(Y_{\Lambda_j}|\mathcal{G})$$

and

$$E(\varphi(Y)|\mathcal{G}) = \sum_{j=1}^n \varphi(y_j) P(Y_{\Lambda_j}|\mathcal{G}).$$

Since $\sum_{j=1}^n P(\Lambda_j|\mathcal{G}) = 1$, the inequality holds by the convexity of φ.

In general we can find a sequence of simple r.v. $\{Y_m\}$ with $|Y_m| \le |Y|$ and $Y_m \to Y$, then apply the above together with the dominated convergence theorem. $\square$

Proposition 1.2.5. *Let Y and YZ be integrable r.v. and $Z \in \mathcal{G}$, then we have*

$$E(YZ|\mathcal{G}) = ZE(Y|\mathcal{G}).$$

Proof. It suffices to show that for $Y, Z \geq 0$

$$\int_\Lambda ZE(Y|\mathcal{G})dP = \int_\Lambda ZY dP \qquad \forall\ \Lambda \in \mathcal{G}.$$

Obviously, this is true for $Z = \chi_\Delta,\ \Delta \in \mathcal{G}$. We can pass it to the simple r.v. Then use the monotone convergence theorem to show that it hold for all $Z \geq 0$, and then the general integrable r.v. $\square$

Proposition 1.2.6. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be sub-σ-fields of $\mathcal{F}$ and $\mathcal{G}_1 \subseteq \mathcal{G}_2$. Then for Y integrable r.v.*

$$E(E(Y|\mathcal{G}_2)|\mathcal{G}_1) = E(Y|\mathcal{G}_1) = E(E(Y|\mathcal{G}_1)|\mathcal{G}_2). \tag{1.2.3}$$

Moreover

$$E(Y|\mathcal{G}_1) = E(Y|\mathcal{G}_2) \quad \textit{iff} \quad E(Y|\mathcal{G}_2) \in \mathcal{G}_1. \tag{1.2.4}$$

Proof. Let $\Lambda \in \mathcal{G}_1$, then $\Lambda \in \mathcal{G}_2$. Hence

$$\int_\Lambda E(E(X|\mathcal{G}_2)|\mathcal{G}_1)dP = \int_\Lambda E(Y|\mathcal{G}_2)dP = \int_\Lambda Y dP = \int_\Lambda E(Y|\mathcal{G}_1)dP,$$

and the first identity in (1.2.3) follows. The second identity is by $E(Y|\mathcal{G}_1) \in \mathcal{G}_2$ (recall that $Z \in \mathcal{G}$ implies $E(Z|\mathcal{G}) = Z$).

For the last part, the necessity is trivial, and the sufficiency follows from the first identity. $\square$

As a simple consequence, we have

Corollary 1.2.7. $E(E(Y|X_1, X_2)|X_1) = E(Y|X_1) = E(E(Y|X_1)|X_1, X_2)$.

Exercises

1. (Bayes' rule) Let $\{\Lambda_n\}$ be a $\mathcal{F}$-measurable partition of Ω and let $E \in \mathcal{F}$ with $P(E) > 0$. Then

$$P(\Lambda_n|E) = \frac{P(\Lambda_n)\ P(E|\Lambda_n)}{\sum_n P(\Lambda_n)P(E|\Lambda_n)}\ .$$

2. If the random vector (X, Y) has probability density $p(x, y)$ and X is integrable, then one version of $E(X|X + Y = z)$ is given by

$$\int xp(x, z - x)dx \ / \int p(x, z - x)dx\ .$$

3. Let X be a r.v. such that $P(X > t) = e^{-t},\ t > 0$. Compute $E(X|X \vee t)$ and $E(X|X \wedge t)$ for $t > 0$. (Here $\vee$ and $\wedge$ mean maximum and minimum respectively.

4. If X is an integrable r.v., Y is a bounded r.v., and $\mathcal{G}$ is a sub-σ-field, then

$$E\big(E(X|\mathcal{G})Y\big) = E\big(XE(Y|\mathcal{G})\big).$$

5. Prove that $\operatorname{var}(E(Y|\mathcal{G})) \le \operatorname{var}(Y)$.

6. Let X, Y be two r.v., and let $\mathcal{G}$ be a sub-σ-field. Suppose

$$E(Y^2|\mathcal{G}) = X^2, \quad E(Y|\mathcal{G}) = X,$$

then $Y = X$ a.e.

7. Give an example that $E(E(Y|X_1)|X_2) \ne E(E(Y|X_2)|X_1)$. (*Hint: it suffices to find an example* $E(X|Y) \ne E(E(X|Y)|X)$ *for* Ω *to have three points*).

1.3 Markov Property

Let A be an index set and let $\{\mathcal{F}_\alpha : \alpha \in A\}$ be family of sub-σ-fields of $\mathcal{F}$. We say that the family of $\mathcal{F}_\alpha$'s are *conditionally independent* relative to $\mathcal{G}$ if for any $\Lambda_i \in \mathcal{F}_{\alpha_i}$ $i = 1, \cdots, n$,

$$P(\bigcap_{j=1}^{n} \Lambda_j|\mathcal{G}) \;=\; \prod_{j=1}^{n} P(\Lambda_i|\mathcal{G}). \tag{1.3.1}$$

Proposition 1.3.1. *For $\alpha \in A$, let $\mathcal{F}^{(\alpha)}$ denote the sub-σ-field generated by $\mathcal{F}_\beta,\ \beta \in A\backslash\{\alpha\}$. Then the family $\{\mathcal{F}_\alpha\}_\alpha$ are conditionally independent relative to $\mathcal{G}$ if and only if*

$$P(\Lambda|\ \mathcal{F}^{(\alpha)} \vee \mathcal{G}) = P(\Lambda|\ \mathcal{G}), \quad \Lambda \in \mathcal{F}_\alpha$$

where $\mathcal{F}^{(\alpha)} \vee \mathcal{G}$ is the sub-σ-field generated by $\mathcal{F}^{(\alpha)}$ and $\mathcal{G}$.

Proof. We only prove the case $A = \{1, 2\}$, i.e.,

$$P(\Lambda|\ \mathcal{F}_2 \vee \mathcal{G}) = P(\Lambda|\ \mathcal{G}), \quad \Lambda \in \mathcal{F}_1. \tag{1.3.2}$$

The general case follows from the same argument. To prove the sufficiency, we assume (1.3.2). To check (1.3.1), let $\Lambda \in \mathcal{F}_1$, then for $M \in \mathcal{F}_2$,

$$\begin{aligned} P(\Lambda \cap M\ |\mathcal{G}) &= E(P(\Lambda \cap M\ |\mathcal{F}_2 \vee \mathcal{G})|\mathcal{G}) \\ &= E(P(\Lambda|\ \mathcal{F}_2 \vee \mathcal{G})\chi_M|\mathcal{G}) \\ &= E(P(\Lambda|\ \mathcal{G})\chi_M|\mathcal{G}) \qquad \text{(by (1.3.2))} \\ &= P(\Lambda|\ \mathcal{G})P(M|\ \mathcal{G}). \end{aligned}$$

Hence $\mathcal{F}_1$ and $\mathcal{F}_2$ are $\mathcal{G}$-independent.

To prove the necessity, suppose (1.3.1) holds, we claim that for $\Delta \in \mathcal{G}$, $\Lambda \in \mathcal{F}_1$ and $M \in \mathcal{F}_2$,

$$\int_{M\cap\Delta} P(\Lambda|\mathcal{G})\ dP = \int_{M\cap\Delta} P(\Lambda|\ \mathcal{F}_2 \vee \mathcal{G})\ dP$$

Since the sets of the form $M \cap \Delta$ generate $\mathcal{G} \vee \mathcal{F}_2$, we have $P(\Lambda|\mathcal{G}) = P(\Lambda|\ \mathcal{F}_2 \vee \mathcal{G})$. i.e., (1.3.2) holds.

The claim follows from the following: let $\Lambda \in \mathcal{F}_1$, $M \in \mathcal{F}_2$, then

$$\begin{aligned} E(P(\Lambda|\mathcal{G})\chi_M|\mathcal{G}) &= P(\Lambda|\mathcal{G})P(M|\mathcal{G}) \\ &= P(\Lambda \cap M\ |\mathcal{G}) \qquad \text{(by (1.3.1))} \\ &= E(P(\Lambda|\mathcal{F}_2 \vee\ \mathcal{G})\chi_M|\mathcal{G}) \qquad \square \end{aligned}$$

Corollary 1.3.2. *Let $\{X_\alpha\}_{\alpha\in A}$ be a family of r.v. and let $\mathcal{F}_\alpha$ be the sub-σ-field generated by X_α. Then the X_α's are independent if and only if for any Borel set B,*

$$P(X_\alpha \in B|\mathcal{F}^{(\alpha)}) = P(X_\alpha \in B).$$

Moreover the above condition can be replaced by: for any integrable $Y \in \mathcal{F}_\alpha$,

$$E(Y|\mathcal{F}^{(\alpha)}) \ = \ E(Y).$$

Proof. The first identity follows from Proposition 1.3.1 by taking $\mathcal{G}$ as the trivial σ-field. The second one follows from an approximation by simple function and use the first identity. $\square$

To consider the Markov property, we first consider an important basic case.

Theorem 1.3.3. *Let $\{X_n\}_{n=1}^\infty$ be a sequence of independent r.v. and each X_n has a distribution μ_n on $\mathbb{R}$. Let $S_n = \sum_{j=1}^n X_j$. Then for $B \in \mathscr{B}$,*

$$P(S_n \in B \mid S_1, \cdots, S_{n-1}) = P(S_n \in B \mid S_{n-1}) = \mu_n(B - S_{n-1})$$

(Hence S_n is independent of $S_1, \cdots, S_{n-2}$ given S_{n-1}.)

Proof. We divide the proof into two steps.

Step 1. We show that

$$P(X_1 + X_2 \in B \mid X_1) = \mu_2(B - X_1)$$

First observe that $\mu_2(B - X_1)$ is in $\mathcal{F}_{X_1}$. Let $\Lambda \in \mathscr{F}_{X_1}$, then $\Lambda = X_1^{-1}(A)$ for some $A \in \mathscr{B}$, and

$$\begin{aligned}
\int_\Lambda \mu_2(B - X_1)\ dP &= \int_A \mu_2(B - x_1)\ d\mu_1(x_1) \\
&= \int_A \Big(\int_{x_1 + x_2 \in B} d\mu_2(x_2) \Big)\ d\mu_1(x_1) \\
&= \iint_{x_1 \in A,\ x_1 + x_2 \in B} d(\mu_1 \times \mu_2)(x_1, x_2) \\
&= P(X_1 \in A,\ X_1 + X_2 \in B) \\
&= \int_\Lambda P(X_1 + X_2 \in B \mid \mathscr{F}_{X_1})\ dP
\end{aligned}$$

This implies that $\mu_2(B - X_1) = P(X_1 + X_2 \in B \mid X_1)$.

Step 2. The second equality in the proposition follows from Step 1 by applying to S_{n-1} and X_n. To prove the first identity, we let $\mu^n = \mu_1 \times \cdots \times \mu_n = \mu^{n-1} \times \mu_n$. Let $B_j \in \mathcal{B}$, $1 \le j \le n-1$, and let $\Lambda = \bigcap_{j=1}^{n-1} S_j^{-1}(B_j) \in \mathcal{F}(S_1, \cdots, S_{n-1})$. We show as in Step 1,

$$\int_\Lambda \mu_n(B - S_{n-1})\ dP = \int_\Lambda P(S_n \in B | S_1, \cdots, S_{n-1})\ dP$$

and the identity $\mu_n(B - S_{n-1}) = P(S_n \in B | S_1, \cdots, S_{n-1})$ follows. □

Definition 1.3.4. *We call a sequence of random variables $\{X_n\}_{n=0}^\infty$ a (discrete time) stochastic process. It is called a* Markov process *(*Markov chain *if the state space is countable or finite) if for any n and $B \in \mathcal{B}$,*

$$P(X_{n+1} \in B | X_0, \cdots, X_n) = P(X_{n+1} \in B | X_n).$$

Let $I \subseteq \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and let $\mathcal{F}_I$ denote the sub-σ-field generated by $\mathcal{F}_n,\ n \in I$. Typically, $I = \{n\}$, or $[0,n]$, or (n,∞); $\mathcal{F}_{\{n\}}$ denotes the events at the present, $\mathcal{F}_{[0,n]}$ denotes the events from the past up to the present, and $\mathcal{F}_{(n,\infty)}$ denotes the events in the future. The above Markov property means the future depends on the present and is independent of the past.

One of the most important examples of Markov process is the sequence $\{S_n\}_{n=0}^{\infty}$ in Theorem 1.2.3.

Theorem 1.3.5. *Let $\{X_n\}_{n=0}^{\infty}$ be a stochastic process, then the following are equivalent:*

(a) $\{X_n\}_{n=0}^{\infty}$ has the Markov property;

(b) $P(M|\mathcal{F}_{[0,n]}) = P(M|X_n)$ for all $n \in \mathbb{N}$ and $M \in \mathcal{F}_{(n,\infty)}$;

(c) $P(M_1 \cap M_2\ |X_n) = P(M_1|X_n)\ P(M_2|X_n)$ for all $M_1 \in \mathcal{F}_{[0,n]},\ M_2 \in \mathcal{F}_{(n,\infty)}$ and $n \in \mathbb{N}$.

The conditions remain true if $\mathcal{F}_{(n,\infty)}$ is replaced by $\mathcal{F}_{[n,\infty)}$ (Exercise). Condition (c) can be interpreted as conditioning on the present, the past and the future are independent.

Proof. $(b) \Rightarrow (c)$. Let $Y_i = \chi_{M_i}$ with $M_1 \in \mathcal{F}_{[0,n]},\ M_2 \in \mathcal{F}_{(n,\infty)}$, then

$$\begin{aligned} P(M_1|X_n)\ P(M_2|X_n) &= E(Y_1|X_n)\ E(Y_2|X_n) = E(Y_1 E(Y_2|X_n)|X_n) \\ &= E(Y_1 E(Y_2|\mathcal{F}_{[0,n]})|X_n) = E(E(Y_1Y_2|\mathcal{F}_{[0,n]})|X_n) \\ &= E(Y_1Y_2|X_n) = P(M_1 \cap M_2\ |X_n). \end{aligned}$$

$(c) \Rightarrow (b)$. Let $\Lambda \in \mathcal{F}_{[0,n]}$ be the test set, and let $Y_1 = \chi_\Lambda,\ Y_2 = \chi_M \in$

$\mathcal{F}_{(0,\infty)}$. Then

$$\begin{aligned}\int_\Lambda P(M|X_n)\,dP &= E\big(Y_1E(Y_2|X_n)\big) = E\big(E(Y_1E(Y_2|X_n))|X_n\big)\\ &= E\big(E(Y_1|X_n)E(Y_2|X_n)\big) = E\big(E(Y_1Y_2|X_n)\big)\\ &= \int_\Omega P(\Lambda\cap M|X_n)\,dP = P(\Lambda\cap M).\end{aligned}$$

This implies $P(M|X_n) = P(M|\mathcal{F}_{[0,n]})$.

$(b) \Rightarrow (a)$ is trivial.

$(a) \Rightarrow (b)$. We claim that for each n,

$$E(Y|\mathcal{F}_{[0,n]}) = E(Y|\ X_n) \qquad \forall\ \ Y \in \mathcal{F}_{[n+1,n+k]},\quad k = 1, 2, \cdots . \tag{1.3.3}$$

This will establish (b) for $M \in \bigcup_{k=1}^\infty \mathcal{F}_{(n,n+k)}$; this family of M generates $\mathcal{F}_{(0,\infty)}$.

Note that the Markov property implies (1.3.3) is true for $k = 1$. Suppose the statement is true for k, we consider $Y = Y_1Y_2 \in \mathcal{F}_{[n+1,n+k+1]}$, where $Y_1 \in \mathcal{F}_{[n+1,n+k]}$ and $Y_2 \in \mathcal{F}_{n+k+1}$. Then

$$\begin{aligned}E(Y|\mathcal{F}_{[0,n]}) &= E(\ E(Y|\mathcal{F}_{[0,n+k]})\ |\ \mathcal{F}_{[0,n]})\\ &= E(\ Y_1E(Y_2|\mathcal{F}_{[0,n+k]})\ |\ \mathcal{F}_{[0,n]})\\ &= E(\ Y_1E(Y_2|\mathcal{F}_{n+k})\ |\ \mathcal{F}_{[0,n]}) \qquad \text{(by Markov)}\\ &= E(\ Y_1E(Y_2|\mathcal{F}_{n+k})\ |\ \mathcal{F}_{n}) \qquad \text{(by induction)}\\ &= E(\ Y_1E(Y_2|\mathcal{F}_{[n,n+k]})\ |\ \mathcal{F}_{[0,n]}) \qquad \text{(by Markov)}\\ &= E(\ E(Y_1Y_2|\mathcal{F}_{[n,n+k]})\ |\ \mathcal{F}_{[0,n]})\\ &= E(Y_1Y_2|\mathcal{F}_n)\\ &= E(Y|\mathcal{F}_n).\end{aligned}$$

This implies the inductive step for $Y = \chi_{M_1\cap M_2} = \chi_{M_1}\chi_{M_2}$ with $M_1 \in \mathcal{F}_{[n+1,n+k]}$ and $M_2 \in \mathcal{F}_{n+k+1}$. But the class of all such Y generates $\mathcal{F}_{[n+1,n+k]}$. This implies the claim and completes the proof of the theorem. $\square$

The following random variable plays a central role in stochastic process.

Definition 1.3.6. *A r.v.* $\alpha : \Omega \to \mathbb{N}_0 \cup \{\infty\}$ *is called a* stopping time *(or Markov time or optional r.v.) with respect to* $\{X_n\}_{n=0}^{\infty}$ *if*

$$\{\omega : \alpha(\omega) = n\} \in \mathcal{F}_{[0,n]} \qquad \textit{for each} \quad n \in \mathbb{N}_0 \cup \{\infty\}.$$

It is easy to see the definition can be replaced by $\{\omega : \alpha(\omega) \leq n\} \in \mathcal{F}_{[0,n]}$. In practice, the most important example is: for a given $A \in \mathcal{B}$, let

$$\alpha_A(\omega) = \min\{n \geq 0 : \ X_n(\omega) \in A\}.$$

($\alpha_A(\omega) = \infty$ if $X_n(\omega) \notin A$ for all n.) This is the r.v. of the first time the process $\{X_n\}_{n=0}^{\infty}$ enters A. It is clear that

$$\{\omega : \ \alpha_A(\omega) = n\} = \bigcap_{j=0}^{n-1}\{\omega : \ X_j(\omega) \in A^c, \ X_n(\omega) \in A\} \in \mathcal{F}_{[0,n]},$$

and similarly for $n = \infty$. Hence α_A is a stopping time.

Very often α represents the random time that a specific event happens, and $\{X_{\alpha+n}\}_{n=1}^{\infty}$ is the process after the event has occurred. We will use the following terminologies:

- The pre-α field $\mathcal{F}_\alpha$ is the sets $\Lambda \in \mathcal{F}_{[0,\infty)}$ of the form

$$\Lambda = \bigcup_{0 \leq n \leq \infty} \{\{\alpha = n\} \cap \Lambda_n\}, \qquad \Lambda_n \in \mathcal{F}_{[0,n]}. \tag{1.3.4}$$

  It follows that $\Lambda \in \mathcal{F}_\alpha$ if and only if $\{\alpha = n\} \cap \Lambda \in \mathcal{F}_n$ for each n.

- The post α-process is $\{X_{\alpha+n}\}_{n=1}^{\infty}$ where $X_{\alpha+n}(\omega) = X_{\alpha(\omega)+n}(\omega)$. The post-$\alpha$ field $\mathcal{F}'_\alpha$ is the sub-σ-field generated by the post-α process.

Proposition 1.3.7. *Let $\{X_n\}_{n=0}^{\infty}$ be a stochastic process and let α be a stopping time. Then $\alpha \in \mathcal{F}_\alpha$ and $X_\alpha \in \mathcal{F}_\alpha$.*

Proof. For α to be $\mathcal{F}_\alpha$-measurable, we need to show that $\{\alpha = k\} \in \mathcal{F}_\alpha$. This follows from (1.3.4) by taking $\Lambda_n = \emptyset$ for $n \neq k$ and $\Lambda_k = \Omega$.

That $X_\alpha \in \mathcal{F}_\alpha$ follows from

$$\{\omega : \ X_\alpha(\omega) \in B\} = \bigcup_n \{\omega : \alpha(\omega) = n, \ X_n(\omega) \in B\} \in \mathcal{F}_\alpha$$

for any Borel set $B \in \mathcal{B}$. $\square$

Theorem 1.3.8. *Let $\{X_n\}_{n=0}^{\infty}$ be a Markov-process and α is an a.e. finite stopping time, then for each $M \in \mathcal{F}'_\alpha$,*

$$P(M|\mathcal{F}_\alpha) = P(M| \ \alpha, X_\alpha). \tag{1.3.5}$$

We call this property the *strong* Markov-property.

Proof. Note that the generating sets of $\mathcal{F}'_\alpha$ are $M = \bigcap_{j=1}^{l} X_{\alpha+j}^{-1}(B_j), \ B_j \in \mathcal{B}$. Let $M_n = \bigcap_{j=1}^{l} X_{n+j}^{-1}(B_j) \in \mathcal{F}_{(n,\infty)}$, We claim that

$$P(M| \ \alpha, X_\alpha) = \sum_{n=1}^{\infty} P(M_n|X_n)\chi_{\{\alpha=n\}}. \tag{1.3.6}$$

Indeed if we consider $P(M_n|X_n) = \varphi_n(X_n)$, then it is clear $\sum_{n=1}^{\infty} \varphi_n(X_n)\chi_{\{\alpha=n\}}$ is measurable with respect to the σ-field generated by α and X_α. By making use of Theorem 1.3.5(b), we have

$$\begin{aligned}
\int_{\{\alpha=m, \ X_\alpha \in B\}} \sum_{n=1}^{\infty} P(M_n|X_n)\chi_{\{\alpha=n\}} \, dP &= \int_{\{\alpha=m, \ X_m \in B\}} P(M_m|X_m) \, dP \\
&= \int_{\{\alpha=m, \ X_m \in B\}} P(M_m|\mathcal{F}_{[0,m]}) \, dP \\
&= P(\{\alpha = m, \ X_m \in B\} \cap M_m) \\
&= P(\{\alpha = m, X_\alpha \in B\} \cap M).
\end{aligned}$$

(The last equality is due to $M_m \cap \{\alpha = m\} = M \cap \{\alpha = m\}$). Hence the claim follows.

Now to prove the theorem, let $\Lambda \in \mathcal{F}_\alpha,\ \Lambda = \bigcup_{n=0}^{\infty}(\{\alpha = n\} \cap \Lambda_n)$, then

$$\begin{aligned}
P(\Lambda \cap M) &= \sum_{n=0}^{\infty} P(\{\alpha = n, \Lambda_n\} \cap M_n) \\
&= \sum_{n=0}^{\infty} \int_{\{\alpha=n\}\cap\Lambda_n} P(M_n|\mathcal{F}_{[0,n]})\ dP \\
&= \sum_{n=0}^{\infty} \int_{\Lambda} P(M_n|X_n)\chi_{\{\alpha=n\}}\ dP \quad \text{(by Theorem 1.3.5(b))} \\
&= \int_{\Lambda} P(M_n|\ \alpha, X_\alpha)\ dP \qquad \text{(by (1.3.6))}.
\end{aligned}$$

The theorem follows from this. $\square$

We remark that when α is the constant n, then we can omit the α in (1.3.5) and it reduces to the Markov property as in Theorem 1.3.5. Also if the process is homogeneous (i.e., invariant on the time n), then we can omit the α there. It is because in (1.3.6), the right side can be represented as $\sum_{n=1}^{\infty} \varphi(X_n)\chi_{\{\alpha=n\}}$ (instead of $\varphi_n(X_n)$) which is $\mathcal{F}_\alpha$-measurable. In this case we can rewrite (1.3.5) as

$$P(X_{\alpha+1} \in B|\mathcal{F}_\alpha) = P(X_{\alpha+1} \in B|X_\alpha) \qquad \forall\ B \in \mathcal{B},$$

a direct analog of the definition of Markov property.

There is a constructive way to obtain Markov processes. For a Markov chain $\{X_n\}_{n=0}^{\infty}$, we mean a stochastic process that has a state space $S = \{a_1, a_2, \cdots, a_N\}$ (finite or countable) and a transition matrix

$$P = \begin{pmatrix} p_{11} & \cdots & p_{1N} \\ \vdots & \cdots & \vdots \\ p_{N1} & \cdots & p_{NN} \end{pmatrix}$$

where $p_{ij} \geq 0$ and the row sum is 1; the p_{ij} is the probability from i to j. Suppose the process starts at X_0 with initial distribution $\mu = (\mu_1, \cdots, \mu_N)$, let X_n denote the location of the chain at the n-th time according to the transition matrix P, then $\{X_n\}_{n=0}^{\infty}$ satisfies the Markov property:

$$P(X_{n+1} = x_{n+1} | X_0 = x_0, \cdots, X_n = x_n) = P(X_{n+1} = x_{n+1} | X_n = x_n) = p_{ij}.$$

Also it follows that

$$\begin{aligned} & P(X_0 = x_0,\ X_1 = x_1, \cdots,\ X_n = x_n) \\ = & P(X_0 = x_0) P(X_1 = x_1 | X_0 = x_0) \cdots P(X_n = x_n | X_{n-1} = x_{n-1}) \\ = & \mu_{x_0} p_{x_0 x_1} \cdots p_{x_{n-1} x_n}. \end{aligned}$$

More generally, we consider the state space to be $\mathbb{R}$. Let $\mu : \mathbb{R} \times \mathcal{B} \to [0, 1]$ satisfies

(a) for each x, $\mu(x, \cdot)$ is a probability measure;

(b) for each B, $\mu(\cdot, B)$ is a Borel measurable function.

Let $\{X_n\}_{n=0}^{\infty}$ be a sequence of r.v. with finite dimensional joint distributions $\mu^{(n)}$ for $X_0, \cdots, X_n$ given by

$$\begin{aligned} P(\bigcap_{j=0}^{n} \{X_j \in B_j\}) &= \mu^{(n)}(B_0 \times \cdots \times B_n) \\ &:= \int \cdots \int_{B_0 \times \cdots \times B_n} \mu_0(dx_0) \mu(x_0, dx_1) \cdots \mu(x_{n-1}, dx_n). \end{aligned}$$

where μ_0 is the distribution function of X_0.

It is direct to check from definition that

$$P(X_{n+1} \in B | X_n) = \mu(X_n, B),$$

i.e.,

$$P(X_{n+1} \in B | X_n = x) \;=\; \mu(x, B).$$

Hence $\mu(x, B)$ represents the probability that in the $(n+1)$-step the chain is in B, starting at x in the n-th step. To see that $\{X_n\}_{n=0}^{\infty}$ satisfies the Markov property, we let $\Lambda = \bigcap_{j=0}^{n} \{X_j \in B_j\}$, then

$$\begin{aligned}
\int_{\Lambda} P(X_{n+1} \in B | X_n)\; dP &= \int \cdots \int_{B_0 \times \cdots \times B_n} \mu(x_n, B)\; d\mu^{(n)}(x_0, \cdots, x_n) \\
&= \int \cdots \int_{B_0 \times \cdots B_n \times B} \mu_0(dx_0) \prod_{j=1}^{n+1} \mu(x_{j-1}, dx_j) \\
&= P(\Lambda \cap \{X_{n+1} \in B\}).
\end{aligned}$$

This implies

$$P(X_{n+1} \in B | X_n) = P(X_{n+1} \in B | X_1, \cdots, X_n)$$

and the Markov property follows.

We call the above $\{X_n\}_{n=0}^{\infty}$ a *stationary* (or *homogeneous*) Markov process and $\mu(x, B)$ the transition probability.

Exercises

1. Let $\{X_n\}_{n=0}^\infty$ be a Markov process. Let f be a one-to-one Borel measurable function on $\mathbb{R}$ and let $Y_n = f(X_n)$. Show that $\{Y_n\}_{n=0}^\infty$ is also a Markov process (with respect to the fields generated by $f(X_n)$); but the conclusion does not hold if we do not assume f is one-to-one.

2. Prove the strong Markov property in the form of Theorem 1.3.5(c).

3. If α_1 and α_2 are both stopping times, so are $\alpha_1 \wedge \alpha_2$, $\alpha_1 \vee \alpha_2$ and $\alpha_1 + \alpha_2$. However $\alpha_1 - \alpha_2$ is not necessary a stopping time.

4. Let $\{X_n\}_{n=1}^\infty$ be a sequence of i.i.d.r.v. Let $\{\alpha_k\}_{k=1}^\infty$ be a sequence of strictly increasing finite stopping times. Then $\{X_{\alpha_k+1}\}_{k=1}^\infty$ is also a sequence of i.i.d.r.v. (This is the gambling-system theorem given by Doob).

5. A sequence $\{X_n\}_{n=0}^\infty$ is a Markov chain of *second order* if

$$P(X_{n+1} = j | X_0 = i_0, \cdots, X_n = i_n) = P(X_{n+1} = j | X_{n-1} = i_{n-1}, X_n = i_n).$$

Show that nothing really new is involved because the sequence (X_n, X_{n+1}) is a Markov chain.

6. Let $\mu^{(n)}(x, B)$ be the n-step transition probability in the stationary Markov process. Prove the Chapman-Kolmogorov equation

$$\mu^{(m+n)}(x, B) = \int_{\mathbb{R}} \mu^{(m)}(x, dy)\mu^{(n)}(y, B) \qquad \forall\, m, n \in \mathbb{N}.$$

1.4 Martingales

We first consider a simple example in analysis. Let f be an integrable function on $[0,1]$, let $\mathcal{P}_n = \{0 = \frac{1}{2^n} \leq \cdots \leq \frac{k}{2^n} \cdots \leq 1\}$ be a partition of $[0,1]$ and let $I_{n,k} = [\frac{k}{2^n}, \frac{k+1}{2^n})$. We define the average function f_n of f on the partition $\mathcal{P}_n$:

$$f_n(x) = \sum_{k=0}^{2^{n-1}} a_{n,k}\ \chi_{I_{n,k}}\ , \qquad x \in I_{n,k}. \tag{1.4.1}$$

where $a_{n,k} = \frac{1}{|I_{n,k}|} \int_{I_{n,k}} f(x)dx$. Then $\{f_n\}_n$ converges to f in L^1. Moreover $\{f_n\}_n$ has the following consistency property: for $m > n$

$$f_n(x) = \frac{1}{|I_{n,k}|} \int_{I_{n,k}} f_m(y)dy \qquad x \in I_{n,k}. \tag{1.4.2}$$

This property has been reformulated by Doob in the more general probability setting.

Definition 1.4.1. *Let $\{(X_n, \mathcal{F}_n)\}_{n=1}^{\infty}$ be a sequence of r.v. such that $X_n \in \mathcal{F}_n$. It is called a martingale if*

(a) $\mathcal{F}_n \subset \mathcal{F}_{n+1}$;

(b) $E(|X_n|) < \infty$;

(c) $X_n = E(X_{n+1}|\mathcal{F})$.

It is called a supermartingale (or submartingale) if $\geq$ (or $\leq$ respectively) in (c) holds. We will call $\{X_n\}_n$ a s-martingale if it is any one of the three cases.

Condition (c) can be strengthened as $X_n = E(X_m|\mathcal{F}_n)$ for $m > n$. It follows from

$$E(X_m|\mathcal{F}_n) = E(E(X_m|\mathcal{F}_{m-1})|\mathcal{F}_n) = E(X_{m-1}|\mathcal{F}_n) = \cdots = E(X_n|\mathcal{F}_n) = X_n\ .$$

Martingale has its intuitive background in gambling. If X_n is interpreted as the gambler's capital at time n, then the defining property says that his expected capital after next game, played with the knowledge of the entire past and present, is exactly equal to his current capital. In other words, his expected gain is zero, and is in this sense the game is said to be "fair". The supermartingale and submartingale can be interpreted similarly.

Example 1. As a direct analog of the above function case, we let X be an integrable r.v. and let $\{\mathcal{F}_n\}_{n=1}^{\infty}$ be an increasing sequence of sub-σ-fields (e.g., take $\mathcal{F}_n$ to be a partition). Let $X_n = E(X|\mathcal{F}_n)$. Then $\{X_n\}_{n=1}^{\infty}$ is a martingale. Indeed we see that

$$E(|X_n|) = E(|E(X|\mathcal{F}_n)|) \leq E(E(|X||\mathcal{F}_n)) = E(|X|) < \infty$$

and (b) follows. For (c), we observe that

$$E(X_{n+1}|\mathcal{F}_n) = E(E(X|\mathcal{F}_{n+1})|\mathcal{F}_n) = E(X|\mathcal{F}_n) = X_n.$$

Example 2. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent integrable r.v. with mean zero. Let $S_n = \sum_{j=1}^{n} X_n$ and $\mathcal{F}_n = \mathcal{F}(X_1, \cdots, X_n)$. Then

$$\begin{aligned} E(S_{n+1}|\mathcal{F}_n) &= E(S_n + X_{n+1}|\mathcal{F}_n) \\ &= S_n + E(X_{n+1}|\mathcal{F}_n) \\ &= S_n + E(X_{n+1}) \\ &= S_n. \end{aligned}$$

Hence $\{(S_n, \mathcal{F}_n)\}$ is a martingale.

Proposition 1.4.2. *If $\{(X_n, \mathcal{F}_n)\}_{n=1}^{\infty}$ is a submartingale, and φ is increasing and convex in $\mathbb{R}$. If $\{\varphi(X_n)\}$ is integrable, then $\{(\varphi(X_n), \mathcal{F}_n)\}$ is also a submartingale.*

Proof. Since $X_n \leq E(X_{n+1}|\mathcal{F}_n)$, by the property of φ, we have

$$\varphi(X_n) \leq \varphi(E(X_{n+1}|\mathcal{F}_n)) \leq E(\varphi(X_{n+1})|\mathcal{F}_n) \qquad \Box$$

It follows that if $\{X_n\}_{n=0}^{\infty}$ is a martingale (or submartingale), then $\{|X_n|^p\}_{n=0}^{\infty}$, $p \geq 1$ (provided that $X_n \in L^p$) and $\{X_n^+\}_{n=0}^{\infty}$ are submartingales. Also if $\{X_n\}$ is a supermartingale, so does $\{X_n \wedge a\}_n$ for any $a \in \mathbb{R}$.

Theorem 1.4.3. *(Doob's decomposition Theorem) For any submartingale $\{(X_n, \mathcal{F}_n)\}_{n=1}^{\infty}$, X_n can be decomposed as*

$$X_n = Y_n + Z_n$$

where $\{(Y_n, \mathcal{F}_n)\}_{n=1}^{\infty}$ is a martingale and $\{Z_n\}$ is a non-negative increasing process.

Proof. We define the difference r.v.

$$D_1 = X_1, \quad D_j = X_j - X_{j-1}, \quad j \geq 2.$$

Then $X_n = \sum_{j=1}^{n} D_j$, and the defining relation of submartingale yields

$$E(D_j|\mathcal{F}_{j-1}) \geq 0, \quad j \geq 2. \tag{1.4.3}$$

We consider yet another difference

$$S_1 = D_1, \quad S_j = D_j - E(D_j|\mathcal{F}_{j-1}),$$

and let

$$Y_n = \sum_{j=1}^{n} S_j, \qquad Z_n = \sum_{j=1}^{n} E(D_j|\mathcal{F}_{j-1}).$$

It is clear that $X_n = Y_n + Z_n$, $X_1 = Y_1$, $Z_1 = 0$ and $\{Z_n\}_{n=1}^{\infty}$ is a non-negative increasing process (by (1.4.3)). On the other hand, note that $E(S_j|\mathcal{F}_{j-1}) = 0$, it follows that

$$E(Y_n|\mathcal{F}_{n-1}) = \sum_{j=1}^{n-1} S_j = Y_{n-1}$$

and hence a martingale. $\square$

For an increasing family of sub-σ-fields $\{\mathcal{F}_n\}_{n=1}^{\infty}$, let $\mathcal{F}_\infty = \bigcup_{n=1}^{\infty} \mathcal{F}_n$ and let α be a stoping time with respect to $\{\mathcal{F}_n\}_{n=1}^{\infty}$, i.e.,

$$\alpha : \Omega \to \mathbb{N} \cup \{\infty\} \quad \text{such that} \quad \{\alpha = n\} \in \mathcal{F}_n$$

As in last section, the pre-α field $\mathcal{F}_\alpha$ is the family of sets

$$\Lambda = \bigcup_n (\{\alpha = n\} \cap \Lambda_n), \quad \Lambda_n \in \mathcal{F}_n.$$

The following theorems aim at replacing the constant time of a martingale by a stoping time.

Theorem 1.4.4. *Let Y be integrable r.v. and let $X_n = E(Y|\mathcal{F}_n)$ where $\mathcal{F}_n$ is an increasing family of sub-σ-fields (it is a martingale). Then for any stopping time α, we have $X_\alpha = E(Y|\mathcal{F}_\alpha)$.*

Moreover if β is also a stopping time and $\alpha \le \beta$, then $\{(X_\alpha, \mathcal{F}_\alpha),\ (X_\beta, \mathcal{F}_\beta)\}$ is a two term martingale (i.e., $X_\alpha = E(X_\beta|\mathcal{F}_\alpha)$).

Proof. Note that $X_\alpha \in \mathcal{F}_\alpha$. We claim that it is also integrable. Indeed as

$$|X_n| = |E(Y|\mathcal{F}_n)| \le E(|Y||\mathcal{F}_n),$$

we have

$$\int_\Omega |X_\alpha| dP = \sum_n \int_{\{\alpha=n\}} |X_n| dP \le \sum_n \int_{\{\alpha=n\}} |Y| dP = \int_\Omega |Y| dP < \infty.$$

Now if $\Lambda \in \mathcal{F}_\alpha$, let $\Lambda_n = \Lambda \cap \{\alpha = n\}$, then

$$\int_\Lambda X_\alpha dP = \sum_n \int_{\Lambda_n} X_n dP = \sum_n \int_{\Lambda_n} Y dP = \int_\Lambda Y dP\ .$$

Hence $X_\alpha = E(Y|\mathcal{F}_\alpha)$.

For the last statement, note that $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$, hence

$$E(X_\beta|\mathcal{F}_\alpha) = E(E(Y|\mathcal{F}_\beta)|\mathcal{F}_\alpha) = X_\alpha. \qquad \square$$

Corollary 1.4.5. *Under the above assumption and suppose* $\{\alpha_i\}_{i=1}^{\infty}$ *is an increasing sequence of stopping times. If* $\{(X_n, \mathcal{F}_n)\}_n$ *is an s-martingale, then* $\{(X_{\alpha_i}, \mathcal{F}_{\alpha_i})\}_i$ *is an s-martingale.*

Unlike Theorem 1.4.4, in the following theorem, we do not assume that the $\{X_n\}$ is the conditional expectation of an integrable Y.

Theorem 1.4.6. *Let* $\{(X_n, \mathcal{F}_n)\}_n$ *be a s-martingale. Let* $\alpha \leq \beta$ *be two bounded stopping times, then* $\{(X_\alpha, \mathcal{F}_\alpha), (X_\beta, \mathcal{F}_\beta)\}$ *is also an s-martingale (of the same type).*

Proof. We prove the theorem for supermartingale. For submartigale, we consider $\{-X_n\}$ instead.

Let $\Lambda \in \mathcal{F}_\alpha$, and let $\Lambda_j = \Lambda \cap \{\alpha = j\}$ $(\in \mathcal{F}_j)$. Then for $k \geq j$, $\Lambda_j \cap \{\beta > k\} \in \mathcal{F}_k$, hence

$$\begin{aligned}\int_{\Lambda_j \cap \{\beta \geq k\}} X_k dP &= \int_{\Lambda_j \cap \{\beta > k\}} X_k dP + \int_{\Lambda_j \cap \{\beta = k\}} X_k dP \\ &\geq \int_{\Lambda_j \cap \{\beta > k\}} X_{k+1} dP + \int_{\Lambda_j \cap \{\beta = k\}} X_k dP\end{aligned}$$

i.e.,

$$\int_{\Lambda_j \cap \{\beta \geq k\}} X_k dP - \int_{\Lambda_j \cap \{\beta \geq k+1\}} X_{k+1} dP \geq \int_{\Lambda_j \cap \{\beta = k\}} X_\beta dP$$

Summing over k, $j \leq k \leq m$, where m is the upper bound of β, then

$$\int_{\Lambda_j \cap \{\beta \geq j\}} X_\alpha dP - \int_{\Lambda_j \cap \{\beta \geq m+1\}} X_{m+1} dP \geq \int_{\Lambda_j \cap \{j \leq \beta \leq m\}} X_\beta dP$$

Hence

$$\int_{\Lambda_j} X_\alpha dP \geq \int_{\Lambda_j} X_\beta dP$$

Summing over $1 \leq j \leq m$,we have

$$\int_\Lambda X_\alpha dP \geq \int_\Lambda X_\beta dP \qquad \forall\, \Lambda \in \mathcal{F}_\alpha. \qquad \square$$

Corollary 1.4.7. *If $\{(X_n, \mathcal{F}_n)\}$ is a martingale or a supermartingale, then the same is for $\{(X_{\alpha\wedge n}, \mathcal{F}_{\alpha\wedge n})\}$ for any stopping time α.*

The theorem still holds if α, β are unbounded. For this we need to associate a random variable X_∞ at ∞.

Theorem 1.4.8. *Assume $\lim_{n\to\infty} X_n = X_\infty$ exists and is integrable. Let α, β be two arbitrary stopping times. Then Theorem 1.4.6 still hold if $\{(X_n, \mathcal{F}_n)\}_{n\in N_\infty}$ is a supermartingale.*

Proof. We first assume that $X_n \geq 0$ and $X_\infty = 0$. Then $X_\alpha \leq \liminf_{n\to\infty} X_{\alpha\wedge n}$, and hence X_α is integrable by Fatou's lemma. The same is for X_β.

From the proof of Theorem 1.4.6, we can conclude that for any m

$$\int_{\Lambda\cap\{\alpha=j\}} X_\alpha dP \geq \int_{\Lambda\cap\{\alpha=j\}\cap\{\beta\leq m\}} X_\beta dP \ .$$

By letting $m \to \infty$ and summing over all j, we have

$$\int_{\Lambda\cap\{\alpha<\infty\}} X_\alpha dP \geq \int_{\Lambda\cap\{\beta<\infty\}} X_\beta dP \ .$$

In addition we have $X_\alpha = X_\infty = 0$ on $\{\alpha = \infty\}$, and $X_\beta = X_\infty = 0$ on $\{\beta = \infty\}$, We conclude that

$$\int_\Lambda X_\alpha dP = \int_\Lambda X_\beta dP$$

and hence $\{(X_\alpha, \mathcal{F}_\alpha),\ (X_\beta, \mathcal{F}_\beta)\}$ is a supermartingale.

For the general case we let

$$X'_n = E(X_\infty|\mathcal{F}_n), \quad X''_n = X_n - X'_n.$$

Then $\{X'_n\}$ is a martingale, and $X_n \geq X'_n$ by the defining property of supermartingale apply to X_n and X_∞. We can apply the above proved case to X''_n, and conclude that $\{X_n\}$ is a supermartingale. $\square$

The above theorems are referred as Doob's *optional sampling theorems.* In terms of gambling, one would hope to devise a strategy to gain advantage of the outcome, but the theorems say that such a strategy does not exist, at least mathematically. The reader can refer to [1, p.327](and the exercises there) for a discussion of the *gambler's ruin problem.*

We use the above stopping time consideration to prove a useful inequality for sub-martingales.

Theorem 1.4.9. *If $\{(X_j, \mathcal{F}_j)\}_{j=1}^n$ is a submartingale, then for any real λ, we have*

$$\lambda\ P(\max_{1\le j\le n} X_j > \lambda) \ \le\ \int_{\{\max_{1\le j\le n} X_j > \lambda\}} X_n dP \ \le\ E(X_n^+).$$

Proof. Let α be the first j such that $X_j \ge \lambda$ if such $1 \le j \le n$ exists, otherwise let $\alpha = n$. It is clear that α is a stopping time, and hence $\{X_\alpha, X_n\}$ is a submartingale (Theorem 1.4.6). If we write

$$M = \{\max_{1\le j\le n} X_j \ge \lambda\},$$

then $M \in \mathcal{F}_\alpha$ and $X_\alpha \ge \lambda$ on M, hence the first inequality follows from

$$\lambda P(M) \le \int_M X_\alpha dP \le \int_M X_n dP.$$

The second inequality is clear. $\square$.

Corollary 1.4.10. *If $\{(X_n, \mathcal{F}_n)\}_{n=1}^\infty$ is a martingale, then for any $\lambda > 0$, we have*

$$P(\max_{1\le j\le n} |X_j| > \lambda) \le \int_{\{\max_{1\le j\le n} |X_j| > \lambda\}} |X_n| dP \le\ \frac{1}{\lambda} E(|X_n|)\ .$$

In addition if $E(|X_n|^2) < \infty$, then we also have

$$P(\max_{1\le j\le n} |X_j| > \lambda) \ \le\ \frac{1}{\lambda^2} E(|X_n|^2)\ .$$

For a sequence of independent r.v. $\{X_n\}_{n=0}^{\infty}$ with zero mean and finite variance, we let $S_n = \sum_{j=1}^{n} X_j$. It is well known (Kolmogorov's inequality [1, p. 116]) that for any $\lambda > 0$,

$$P(\max_{1\le j\le n} |S_j| > \lambda) \ \le \ \frac{1}{\lambda^2} E(|S_n|^2).$$

We see that the inequality follows directly from the above corollary.

To conclude this section, we prove a deep theorem on the convergence of the $\{X_n\}_n$, which is also due to Doob. It involves an ingenious method in the proof.

Theorem 1.4.11. *If $\{(X_n, \mathcal{F}_n)\}_{n=0}^{\infty}$ is an L^1-bounded submartingale, then $\{X_n\}_{n=0}^{\infty}$ converges a.e. to a finite limit.*

Proof. First we define, for any pair of rationals a, b, let

$$\Lambda_{[a,b]} = \{\omega : \ \liminf_{n\to\infty} X_n(\omega) < a < b < \limsup_{n\to\infty} X_n(\omega)\} \tag{1.4.4}$$

We show that $\Lambda_{[a,b]}$ is a zero set for any $a, b \in \mathbb{Q}$. It follows that

$$\{\omega : \ \liminf_{n\to\infty} X_n(\omega) < \limsup_{n\to\infty} X_n(\omega)\} = \bigcup_{a,b\in Q, \ a<b} \Lambda_{[a,b]}$$

is a zero set. Note that $\liminf_{n\to\infty} X_n$ is finite almost everywhere (by Fatou lemma and the L^1-boundedness assumption, $E(\liminf |X_n|) \le \liminf E(|X_n|) < \infty$), hence the theorem follows.

It remains to prove (1.4.4). We first introduce some notations. Let $\{x_1, \cdots, x_n\}$ be a numerical sequence, for $a < b$, let

$$\alpha_1 = \min\{j : \ 1 \le j \le n, \ x_j \le a\},$$

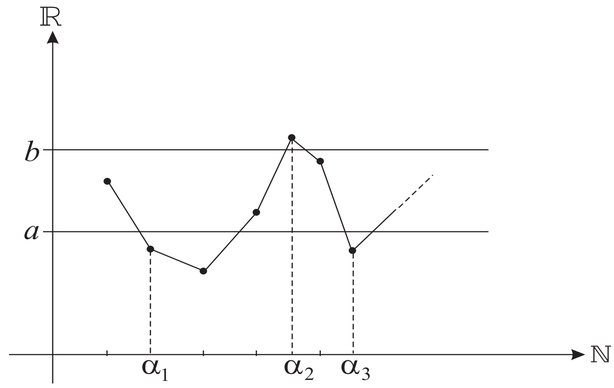
$$\alpha_2 = \min\{j : \ \alpha_1 < j \le n, \ x_j \ge b\}.$$

Inductively we define

$$\alpha_{2k-1} = \min\{j : \ \alpha_{2k-2} < j \le n, \ x_j \le a\},$$

$$\alpha_{2k} = \min\{j : \ \alpha_{2k-1} < j \le n, \ x_j \ge b\}.$$

Let α_l be the last one defined. We can think of connecting the consecutive x_i by line segments, Let ν be the number of times the line segments comes from $\le a$ to $\ge b$, i.e., the number of upcrossing through the interval $[a, b]$, it is seen that $\nu = [l/2]$.



Lemma 1.4.12. *Let* $\{(X_j, \mathcal{F}_j)\}_{j=1}^n$ *be a submartingale and assume that* $X_j \ge 0$. *Let* $\nu_{[0,b]}^{(n)}$ *be the r.v. of the number of upcrossing of* $[0, b]$ *by the sample sequence* $\{X_j(\omega) : \ 1 \le j \le n\}$. *Then*

$$E(\nu_{[0,b]}^{(n)}) \ \le \ \frac{E(X_n - X_1)}{b} \ .$$

Proof. For convenience, we let $\alpha_0 = 1$ and $\alpha_{l+1} = \alpha_{l+2} = \cdots = \alpha_n = n$. Then we have a sequence of stopping times with

$$1 = \alpha_0 \le \alpha_1 < \cdots < \alpha_l \le \alpha_{l+1} \cdots \le \alpha_n = n.$$

We write

$$X_n - X_1 = X_{\alpha_n} - X_{\alpha_0} = \sum_{j=1}^{n-1}(X_{\alpha_{j+1}} - X_{\alpha_j}) = \Big(\sum_{j \ odd} + \sum_{j \ even}\Big)(X_{\alpha_{j+1}} - X_{\alpha_j}).$$

It follows that

$$\sum_{j\ odd} \big(X_{\alpha_{j+1}}(\omega) - X_{\alpha_j}(\omega)\big) \ \geq \ [l(\omega)/2]\cdot b \ = \ \nu^{(n)}_{[0,b]}(\omega)\cdot b.$$

On the other hand by Theorem 1.4.6, $\{X_{\alpha_j} : 0 \leq j \leq n\}$ is a submartingale, so that for each $0 \leq j \leq n-1$, $E(X_{\alpha_{j+1}} - X_{\alpha_j}) \geq 0$. Consequently

$$E(\sum_{j\ even} (X_{\alpha_{j+1}} - X_{\alpha_j})) \geq 0.$$

Therefore $E(X_n - X_1) \geq E(v^{(n)}_{[0,b]})\cdot b$ which yields the lemma. $\square$.

Now to complete the proof of (1.4.4), we consider the upcrossing on any [a,b]. We replace the r.v. in the lemma by $(X_n - a)^+$. The sequence $\{(X_n - a)^+\}_n$ is still submartingale and by the lemma,

$$E(\nu^{(n)}_{[a,b]}) \leq \frac{E(X_n - a)^+ - E(X_1 - a)^+}{b-a} \leq \frac{E({X_n}^+) + |a|}{b-a}.$$

Let $\nu_{[a,b]} = \lim_{n\to\infty} \nu^{(n)}_{[a,b]}$. The L^1-boundedness of $\{X_n\}_n$ implies that $E(\nu_{[a,b]}) < \infty$. Hence $\nu_{[a,b]}$ is finite with probability 1. Note that

$$\begin{aligned} \Lambda_{[a,b]} &= \{\omega : \ \liminf_{n\to\infty} X_n(\omega) \leq a < b \leq \limsup_{n\to\infty} X_n(\omega)\} \\ &\subseteq \{\omega : \ \nu_{[a,b]}(\omega) = \infty\}, \end{aligned}$$

hence $\Lambda_{[a,b]}$ is a zero set and (1.4.4) follows. This completes the proof of the theorem. $\square$

Corollary 1.4.13. *Every uniformly bounded s-martingale converges a.e. Also every positive supermartingale and every negative submartingale converges a.e.*

Proof. The first statement follows directly from Theorem 1.4.8 and that $\{X_n\}$ is a submartingale if and only $\{-X_n\}$ is a supermartingale.

For the second part we use Doob's decomposition theorem (Theorem 1.4.3. Let $\{X_n\}_n$ be a positive supermartingale, then $X_n = Y_n - Z_n$ where $\{Y_n\}$ is a martingale and $Z_n \geq 0$, $\{Z_n\} \nearrow$. Since $X_n \geq 0$, it follows that $0 \leq Z_n \leq Y_n$. Let $Z_\infty = \lim_{n\to\infty} Z_n$. It is finite a.e. because

$$E(Z_\infty) = \lim_{n\to\infty} E(Z_n) \leq E(Y_1) < \infty.$$

Also since $\{X_n\}_n$ is a supermartingale,

$$E(Y_n) = E(X_n) + E(Z_n) \leq E(X_1) + E(Z_\infty).$$

This implies $\{Y_n\}_n$ is L^1-uniformly bounded and $\{Y_n\}_n$ converges to a finite limit a.e. (Theorem 1.4.8). The same holds for $\{X_n\}_n$. $\square$

Recall that a sequence of r.v. $\{X_n\}_{n=1}^\infty$ is called uniformly integrable if

$$\lim_{k\to\infty} \int_{|X_n|\geq k} |X_n| dP = 0 \qquad \text{uniformly on } n\ .$$

It is clear that it implies that $\{X_n\}_{n=1}^\infty$ is L^1- bounded. Also, if $X_n \to X$ a.e., then the uniformly boundedness implies that $X_n \to X$ in L^1 ([1, p.96-97]).

Corollary 1.4.14. *If $\{(X_n, \mathcal{F}_n)\}_{n=1}^\infty$ is a submartingale and is uniformly integrable, then $X_\infty = \lim_{n\to\infty} X_n$ a.e. and in L^1.*

Remark. Theorem 1.4.11 and Corollary 1.4.14 are more or less that the converse of Example 1. However for Example 2, the sum $\{S_n\}_{n=1}^\infty$ of i.i.d.r.v. $\{X_n\}_{n=0}^\infty$ with zero mean forms a martingale, but does not converge; it is because the L^1-bounded condition is not satisfies. In fact, we can show that

$$\lim_{n\to\infty} E\big(\frac{|S_n|}{\sqrt{n}}\big) = \sqrt{\frac{2}{\pi}}\sigma$$

where σ is the variance of X_n. For more detail, the reader can refer to [1, Chapter 5, 6] for the law of large number and the central limit theorem for $\{S_n\}_{n=1}^\infty$.

Exercises

1. Suppose $\{(X_n^{(k)}, \mathcal{F}_n)\}_n,\ k = 1, 2$ are two martingales, α is a finite stopping time and $X_\alpha^{(1)} = X_\alpha^{(2)}$. Define $X_n = X_n^{(1)}\chi_{\{n\leq\alpha\}} + X_n^{(2)}\chi_{\{n\leq\alpha\}}$. Show that $\{(X_n, \mathcal{F}_n)\}_n$ is a martingale.

2 If $\{(X_n, \mathcal{F}_n)\}_n$, $\{(Y_n, \mathcal{F}_n\}_n$ are martingales, then $\{(X_n + Y_n, \mathcal{F}_n)\}$ is again a martingale. However it may happen that $\{X_n\}_n$, $\{Y_n\}_n$ are martingales, but $\{X_n + Y_n\}_n$ is not a martingale. (Note the the σ-field generated by $X_n + Y_n$ may not have the same σ-field $\mathcal{F}_n$.)

3 Prove that for any L^1-bounded s-martingale $\{(X_n, \mathcal{F}_n)\}_n$, and for any α stopping time, then $E(|X_\alpha|) < \infty$.

4. If X is an integrable r.v., then the collection of r.v., $D(X|\mathcal{G})$ with $\mathcal{G}$ ranging over all Borel subfields of $\mathcal{F}$, is uniformly integrable.

5. Find an example of a positive martingale that is not uniformly integrable.

6. Find an example of a martingale $\{X_n\}_n$ such that $X_n \to -\infty$. This implies that in a fair game one player may lose an arbitrary large amount if he stays on long enough. (Hint: Try sums of independent but not identically distributed r.v. with mean 0.)

7. If $\{X_n\}_n$ is a uniformly integrable submartingale, then for any stopping time α, $\{X_{\alpha\wedge n}\}_n$ is again a uniformly integrable submartingale and

$$E(X_1) \leq E(X_\alpha) \leq \sup_n E(X_n).$$

8 Prove that for any s-martingale, we have for each $\lambda > 0$,

$$\lambda P(\sup_n |X_n| \geq \lambda) \ \leq 3 \sup_n E(|X_n|)\ .$$

For a martingale or a positive or nonnegative s-martingale the constant 3 may be replaced by 1.

9. Let $\{X_n\}_n$ be a positive supermartingale. Then for almost every ω, $X_k(\omega) = 0$ implies $X_n(\omega) = 0$ for all $n \geq k$.

10. Every L^1-martingale is the difference of two positive L^1-bounded martingales. (Hint, take one of them to be $\lim_{k\to\infty} E(X_k^+|\mathcal{F}_n)$).

11. If $\{X_n\}$ is a martingale or positive submartingale such that $\sup_n E(X_n^2) \leq \infty$, then $\{X_n\}_n$ converges in L^2 as well as a.e.

12. Show that if $\{(X_n, \mathcal{F}_n)\}_n$ is a submartingate, $X_n \geq 0$, then for $p > 1$,

$$|| \max_{\{1\leq k\leq n\}} ||_p \;\leq\; \frac{p}{p-1} ||X_n||_p \;.$$

(Hint: Show that for $Y \geq 0$, $E(Y^p) = p \int_0^\infty \lambda^{p-1} P(Y \geq \lambda) d\lambda$.)