Lecture 8

4.4 Calculation of entropy

Lemma 4.9. *(i)* $H(T^{-k}\xi|T^{-k}\eta) = H(\xi|\eta)$ *for* $k > 0$.
(ii) $h(T,\xi) \le h(T,\eta) + H(\xi|\eta)$.
(iii) $h(T, \xi \vee T^{-1}\xi \vee \cdots \vee T^{-(n-1)}\xi) = h(T,\xi)$ *for* $n > 0$.

Proof. (i) By the basic identity,

$$\begin{aligned} H(T^{-k}\xi|T^{-k}\eta) &= H(T^{-k}\xi \vee T^{-k}\eta) - H(T^{-k}\eta) \\ &= H(\xi\vee\eta) - H(\eta) = H(\xi|\eta). \end{aligned}$$

(ii) Notice that

$$\begin{aligned} H(\xi \vee T^{-1}\xi \vee \cdots \vee T^{-(n-1)}\xi) &\le H(\eta \vee T^{-1}\eta \vee \cdots \vee T^{-(n-1)}\eta) \\ &\quad + H(\xi \vee T^{-1}\xi \vee \cdots \vee T^{-(n-1)}\xi | \eta \vee T^{-1}\eta \vee \cdots \vee T^{-(n-1)}\eta) \\ &\le H(\eta \vee T^{-1}\eta \vee \cdots \vee T^{-(n-1)}\eta) + \sum_{i=0}^{n-1} H(T^{-i}\xi|\eta \vee T^{-1}\eta \vee \cdots \vee T^{-(n-1)}\eta) \\ &\le H(\eta \vee T^{-1}\eta \vee \cdots \vee T^{-(n-1)}\eta) + \sum_{i=0}^{n-1} H(T^{-i}\xi|T^{-i}\eta) \\ &= H(\eta \vee T^{-1}\eta \vee \cdots \vee T^{-(n-1)}\eta) + nH(\xi|\eta). \end{aligned}$$

Hence

$$\frac{1}{n}H(\xi \vee T^{-1}\xi \vee \cdots \vee T^{-(n-1)}\xi) \le \frac{1}{n}H(\eta \vee T^{-1}\eta \vee \cdots \vee T^{-(n-1)}\eta) + H(\xi|\eta),$$

letting $n \to \infty$, we obtain (ii).

(iii) Set $\eta = \xi \vee T^{-1}\xi \vee \cdots \vee T^{-(n-1)}\xi$, then

$$\begin{aligned} h(T,\eta) &= \lim_{m\to\infty} \frac{1}{m} H\Big(\bigvee_{i=0}^{m-1} T^{-i}\eta\Big) \\ &= \lim_{m\to\infty} \frac{1}{m} H\Big(\bigvee_{i=0}^{m+n-2} T^{-i}\xi\Big) \\ &= \lim_{m\to\infty} \frac{1}{m+n-1} H\Big(\bigvee_{i=0}^{m+n-2} T^{-i}\xi\Big) = h(T,\xi). \end{aligned}$$

□

Proposition 4.2. $h(T^n) = nh(T)$ *for* $n > 0$.

Proof. Let ξ be a finite partition, set $\eta = \xi \vee T^{-1}\xi \vee \cdots \vee T^{-(n-1)}\xi$. Then

$$\begin{aligned} nh(T,\xi) &= \lim_{m\to\infty} \frac{n}{mn} H(\xi \vee T^{-1}\xi \vee \cdots \vee T^{-(mn-1)}\xi) \\ &= \lim_{m\to\infty} \frac{1}{m} H(\eta \vee T^{-n}\eta \vee \cdots \vee T^{-n(m-1)}\eta) \\ &= h(T^n,\eta) \le h(T^n), \end{aligned}$$

taking supremum over ξ, we have $nh(T) \le h(T^n)$. On the other hand, since $\xi \le \eta$,

$$h(T^n,\xi) \le h(T^n,\eta) = nh(T,\xi) \le nh(T),$$

taking supremum over ξ, we have $h(T^n) \le nh(T)$. □

Theorem 4.10. *Let $(X, \mathscr{B}, \mu, T)$ be a MPS. Moreover, X is a compact metric space and $\mathscr{B}$ is the Borel σ-algebra over X. If $\{\xi_n\}_{n=1}^{\infty}$ is a sequence of Borel partitions of X with $diam(\xi_n) := \max_{A\in\xi_n} diam(A) \to 0$ as $n \to \infty$, then*

$$h(T) = \lim_{n\to\infty} h(T,\xi_n).$$

To prove this theorem, we need establish the following lemmas.

Lemma 4.11. *Under the condition of Theorem 4.10, let $\mathscr{C} = \{C_1, \cdots, C_k\}$ be a finite partition of X, then we can find partitions $\{E_1^n, \cdots, E_k^n\}$ with each E_i^n being a union of some elements in ξ_n such that for $i = 1, \cdots, k$,*

$$\mu(C_i \triangle E_i^n) \to 0, \ \text{as } n \to \infty.$$

Proof. Let $\epsilon > 0$, pick compact sets $K_1, \cdots, K_k$ such that $K_i \subset C_i$ and $\mu(C_i \setminus K_i) < \epsilon$. Let $\delta = \inf_{i\neq j} d(K_i, K_j) > 0$. Consider ξ_n with $\text{diam}(\xi_n) < \frac{\delta}{2}$. Since each element of ξ_n can intersect with at most one K_i, we can divide the elements of ξ_n into groups whose union are $E_1^n, \cdots, E_k^n$, so that $B \subset E_i^n$ if $B \cap K_i \neq \emptyset$ for $B \in \xi_n$, for those $B \in \xi_n$ that do not intersect with any K_i, put it into any E_i^n as you like. Then $K_i \subset E_i^n$ for $i = 1, 2, \cdots, k$. Moreover, since if $x \in E_i^n \setminus C_i$, then $x \notin K_i$ and $x \notin K_j$ for $j \neq i$, hence $E_i^n \setminus C_i \subset X \setminus \cup_{j=1}^k K_j$. Then for $i = 1, \cdots, k$ and for all n large we have

$$\begin{aligned} \mu(C_i \triangle E_i^n) &= \mu(C_i \setminus E_i^n) + \mu(E_i^n \setminus C_i) \\ &\le \mu(C_i \setminus K_i) + \mu(X \setminus \cup_{j=1}^k K_j) \\ &\le (k+1)\epsilon. \end{aligned}$$

Hence for $i = 1, \cdots, k$,

$$\overline{\lim_{n\to\infty}} \mu(C_i \triangle E_i^n) \le (k+1)\epsilon,$$

since $\epsilon > 0$ is arbitrary, we complete the proof. □

Lemma 4.12. *Under the assumption of Theorem 4.10. Let* $\mathscr{C} = \{C_1, \cdots, C_k\}$ *be a finite partition. Then*

$$\lim_{n\to\infty} H(\mathscr{C}|\xi_n) = 0.$$

Proof. Using the above lemma we find partitions $\gamma_n = \{E_1^n, \cdots, E_k^n\}$ with each E_i^n being a union of elements in ξ_n, so that

$$\mu(C_i \triangle E_i^n) \to 0 \text{ as } n \to \infty.$$

Since $\gamma_n \leq \xi_n$, we have $H(\mathscr{C}|\xi_n) \leq H(\mathscr{C}|\gamma_n)$. By continuity of ϕ, we have

$$H(\mathscr{C}|\gamma_n) = \sum_{i,j} \mu(E_i^n)\phi\big(\frac{m(C_i \cap E_i^n)}{m(E_i^n)}\big) \to \sum_{i,j} \mu(C_i)\phi\big(\frac{m(C_i \cap C_j)}{m(C_i)}\big) = 0,$$

as $n \to \infty$. This completes the proof. □

Now we can prove Theorem 4.10.

Proof of Theorem 4.10. Let $\mathscr{C}$ be a finite partition of X. Then

$$h(T, \mathscr{C}) \leq h(T, \xi_n) + H(\mathscr{C}|\xi_n) \text{ for } n > 0,$$

letting $n \to \infty$, by the above lemma we have

$$h(T, \mathscr{C}) \leq \varliminf_{n\to\infty} h(T, \xi_n),$$

taking supremum over $\mathscr{C}$, $h(T) \leq \varliminf_{n\to\infty} h(T, \xi_n)$. Since trivially we have $\varlimsup_{n\to\infty} h(T, \xi_n) \leq h(T)$, the limit exists and equals $h(T)$. □

Theorem 4.13. *Let* $(X, \mathscr{B}, \mu, T)$ *be a MPS over a compact metric space. Let* ξ *be a finite partition of* X*. If diam*$\big(\bigvee_{i=0}^{n-1} T^{-i}\xi\big) \to 0$ *as* $n \to \infty$*, then* $h(T) = h(T, \xi)$*.*

Proof. By Theorem 4.10, we have

$$h(T) = \lim_{n\to\infty} h\big(T, \bigvee_{i=0}^{n-1} T^{-i}\xi\big) = \lim_{n\to\infty} h(T, \xi) = h(T, \xi).$$

□

Now we consider some examples.

Example 1. (Rotation on the circle). Let μ be the Haar measure on $\mathbb{R}/\mathbb{Z}$, $Tx := x + \alpha(\text{mod}1)$. Then $h(T) = 0$.

Proof. Case 1. Let $\alpha = \frac{p}{q} \in \mathbb{Q}$, then $T^q =$ identity. Hence for any finite partition ξ,

$$\bigvee_{i=0}^{n-1} T^{-i}\xi = \bigvee_{i=0}^{q-1} T^{-i}\xi, \text{ for } n \geq q.$$

Hence

$$h(T,\xi) = \lim_{n\to\infty} \frac{1}{n} H\big(\bigvee_{i=0}^{n-1} T^{-i}\xi\big) = \lim_{n\to\infty} \frac{1}{n} H\big(\bigvee_{i=0}^{q-1} T^{-i}\xi\big) = 0,$$

for any finite partition ξ, hence $h(T) = 0$.

Case 2. Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$. Let ξ_n be the partition $\{[\frac{j}{n}, \frac{j+1}{n}) : j = 0, 1, \cdots, n-1\}$ of $[0, 1)$. Since $\text{diam}(\xi_n) \to 0$, by Theorem 4.10 $h(T) = \lim_{n\to\infty} h(T, \xi_n)$. We claim $h(T, \xi_n) = 0$ for all n. To see this, notice that $\sharp\big(\bigvee_{i=0}^{m-1} T^{-i}\xi_n\big) \leq mn$, hence $H\big(\bigvee_{i=0}^{m-1} T^{-i}\xi_n\big) \leq \log mn$, then

$$h(T,\xi_n) = \lim_{m\to\infty} \frac{1}{m} H\big(\bigvee_{i=0}^{m-1} T^{-i}\xi_n\big) = 0.$$

$h(T) = 0$ follows. □

Example 2. (Doubling map on the circle). Let μ be the Haar measure on $\mathbb{R}/\mathbb{Z}$, $Tx := 2x(\text{mod}1)$. Then $h(T) = \log 2$.

Proof. Let $\xi = \{[0, \frac{1}{2}), [\frac{1}{2}, 1)\}$, then $\xi \vee T^{-1}\xi \vee \cdots \vee T^{-(n-1)}\xi = \{[\frac{j}{2^n}, \frac{j+1}{2^n}) : j = 0, 1, \cdots, 2^n - 1\}$. Since $\text{diam}(\xi \vee T^{-1}\xi \vee \cdots \vee T^{-(n-1)}\xi) \to 0$ as $n \to \infty$, by Theorem 4.13 $h(T) = h(T, \xi) = \log 2$. □