# Suggested Solution to Assignment 4

## Exercise 4.1

2. The solution to this problem satisfies the following PDE

$$\begin{aligned} &u_t = ku_{xx}, \quad (0 < x < l,\ 0 < t < \infty) \\ &u(0,t) = u(l,t) = 0, \\ &u(x,0) = 1. \end{aligned}$$

Following the process in Page 85 of the textbook, we have

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{l})^2 kt} \sin \frac{n\pi x}{l},$$

and the initial condition implies

$$1 = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}.$$

By the assumption, we have $A_n = \frac{4}{n\pi}$ for odd $n$ and $A_n = 0$ for even ones. Then

$$u(x,t) = \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} e^{-(\frac{(2k-1)\pi}{l})^2 kt} \sin \frac{(2k-1)\pi x}{l}. \qquad \square$$

4. Let $u(x,t) = T(t)X(x)$, we have

$$\frac{T'' + rT'}{c^2 T} = \frac{X''}{X} = -\lambda.$$

Hence,

$$\lambda_n = (\frac{n\pi}{l})^2,\ X(x) = \sin \frac{n\pi x}{l},\ n = 1, 2, \cdots.$$

Since $0 < r < 2\pi c/l$, we get

$$T_n(t) = [A_n \cos(\sqrt{-\Delta_n} t/2) + B_n \sin(\sqrt{-\Delta_n} t/2)] e^{-rt/2}, n = 1, 2, \cdots,$$

where $\Delta_n = r^2 - (2n\pi c/l)^2$ relative to the equation

$$\lambda^2 + r\lambda + (\frac{n\pi c}{l})^2 = 0$$

Therefore ,

$$u(x,t) = \sum_{n=1}^{\infty} [A_n \cos(\sqrt{-\Delta_n} t/2) + B_n \sin(\sqrt{-\Delta_n} t/2)] e^{-rt/2} \sin \frac{n\pi x}{l}. \qquad \square$$

5. Let $u(x,t) = T(t)X(x)$, we have

$$\frac{T'' + rT'}{c^2 T} = \frac{X''}{X} = -\lambda.$$

Hence,

$$\lambda_n = (\frac{n\pi}{l})^2,\ X(x) = \sin \frac{n\pi x}{l},\ n = 1, 2, \cdots.$$

When $n = 1$, since $2\pi c/l < r < 4\pi c/l$,

$$T_1(t) = A_1 e^{\lambda_1^+ t} + B_1 e^{\lambda_1^- t},$$

where $\lambda_1^{\pm} = \dfrac{-r \pm \sqrt{r^2 - (\frac{2\pi c}{l})^2}}{2}$ are the roots of the equation $\lambda^2 + r\lambda + (\frac{\pi c}{l})^2 = 0$.
When $n \geq 2$,

$$T_n(t) = [A_n \cos(\sqrt{-\Delta_n} t/2) + B_n \sin(\sqrt{-\Delta_n} t/2)]e^{-rt/2}, n = 1, 2, \cdots,$$

where $\Delta_n = r^2 - (2n\pi c/l)^2$ relative to the equation $\lambda^2 + r\lambda + (\frac{n\pi c}{l})^2 = 0$.
Therefore ,

$$u(x,t) = [A_1 e^{\lambda_1^+ t} + B_1 e^{\lambda_1^- t}] \sin \frac{\pi x}{l} + \sum_{n=2}^{\infty} [A_n \cos(\sqrt{-\Delta_n} t/2) + B_n \sin(\sqrt{-\Delta_n} t/2)]e^{-rt/2} \sin \frac{n\pi x}{l}. \qquad \square$$

6. Let $u(x,t) = T(t)X(x)$, we have

$$\frac{tT' - 2T}{T} = \frac{X''}{X} = -\lambda,$$

$$\lambda_n = n^2,\ X(x) = \sin nx,\ n = 1, 2, \cdots.$$

The initial condition implies

$$tT' - 2T = -\lambda T,\ T(0) = 0.$$

Therefore,

$$u(x,t) = ct \sin x, \quad \text{for any constant} c,$$

are solutions. So uniqueness is false for this equation! $\square$

## Exercise 4.2

1. Let $u(x,t) = T(t)X(x)$, we have

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda.$$

The initial condition implies

$$-X'' = \lambda X, X(0) = X'(l) = 0.$$

So by solving the above DE, the eigenvalues are $[\frac{(n+\frac{1}{2})\pi}{l}]^2$, the eigenfunctions are $X_n(x) = \sin \frac{(n+\frac{1}{2})\pi x}{l}$ for $n = 0, 1, 2, \cdots$, and the solution is

$$u(x,t) = \sum_{n=0}^{\infty} e^{-[\frac{(n+\frac{1}{2})\pi}{l}]^2 kt} \sin \frac{(n+\frac{1}{2})\pi x}{l}. \qquad \square$$

2. (a) This can be proved as above. Here we give another proof. Since $X'(0) = 0$, the we can use even expansion, this is, $X(-x) = X(x)$ for $-l \leq x \leq 0$, then $X$ satisfies

$$-X'' = \lambda X,\ X(-l) = X(l) = 0.$$

Hence,

$$\lambda_n = [(n + \frac{1}{2})\pi]^2/l^2,\ X_n(x) = \cos[(n + \frac{1}{2})\pi x/l],\ n = 0, 1, 2, \cdots.$$

(b) Having known the eigenvalues, it is easy to get the solution

$$u(x,t) = \sum_{n=0}^{\infty} \left[ A_n \cos \frac{(n+\frac{1}{2})\pi ct}{l} + B_n \sin \frac{(n+\frac{1}{2})\pi ct}{l} \right] \cos \frac{(n+\frac{1}{2})\pi x}{l}. \qquad \square$$

3. We just show how to solve the eigenvalue problem under the periodic boundary conditions; As before, let $u(x,t) = T(t)X(x)$,

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda$$

Solving $T' = -\lambda kT$ gives $T = Ae^{-\lambda kT}$. The general solutions of $X'' + \lambda X = 0$ are $X = Ce^{\gamma x} + De^{-\gamma x}$, where let $\lambda$ is a complex number and $\gamma$ is either one of the two roots of $-\lambda$; the other one is $-\gamma$. The boundary conditions yield

$$Ce^{-\gamma l} + De^{\gamma l} = Ce^{\gamma l} + De^{-\gamma l}, \gamma(Ce^{-\gamma l} - De^{\gamma l}) = \gamma(Ce^{\gamma l} - De^{-\gamma l}).$$

Hence $e^{2\gamma l} = 1$ and then

$$\gamma = \pm n\pi i/l,\ \lambda = -\gamma^2 = (n\pi/l)^2,\ n = 0, 1, 2, \cdots$$
$$X_n(x) = \begin{cases} \frac{1}{2}A_0 & n = 0 \\ A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}, T = e^{-(n\pi/l)^2 kt} & n = 1, 2, \cdots \end{cases}.$$

Therefore, the concentration is

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=0}^{\infty}\left(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}\right)e^{-(n\pi/l)^2 kt}. \qquad \square$$

## Exercise 4.3

1. Firstly, let's look for the positive eigenvalues $\lambda = \beta^2 > 0$. As usual, the general solution of the ODE is

$$X(x) = C\cos\beta x + D\sin\beta x.$$

The boundary conditions imply

$$C = 0,\ D\beta\cos(\beta l) + aD\sin(\beta l) = 0.$$

Hence, $\tan(\beta l) = -\frac{\beta}{a}$. The graph is omitted.

Seconddly, let's look for the zero eigenvalue, i.e., $X(x) = Ax + B$, by the boundary conditions, $al + 1 = 0$. Hence, $\lambda = 0$ is an eigenvalue if and only if $al + 1 = 0$.

Thirdly, let's look for the negative eigenvalues $\lambda = -\gamma^2 < 0$. As usual, the solution of the ODE is

$$X(x) = C\cosh(\gamma x) + D\sinh(\gamma x).$$

Then the boundary condtions imply

$$C = 0,\ D\gamma\cosh(\gamma l) + aD\sinh(\gamma l) = 0.$$

Hence, $\tanh(\gamma l) = -\frac{\gamma}{a}$. The graph is omitted. $\square$

2. (a) If $\lambda = 0$, then $X(x) = Ax + B$. The boundary conditions imply

$$A - a_0 B = 0,\ A + a_l(Al + B) = 0.$$

These two equalities are equivalent to

$$a_0 + a_l = -a_0 a_l l.$$

Hence, $\lambda = 0$ is an eigenvalue if and only if $a_0 + a_l = -a_0 a_l l$.

(b) By (a), we have $X(x) = B(a_0 x + 1)$, here $B$ is constant. $\square$

3. If $\lambda = -\gamma^2 < 0$, we have
$$X(x) = C\cosh\gamma x + D\sinh\gamma x.$$
Hence,
$$X'(x) = C\gamma\sinh\gamma x + D\gamma\cosh\gamma x,$$
and the boundary conditions imply
$$\begin{aligned} D\gamma - a_0 C = 0, \\ C\gamma\sinh\gamma l + D\gamma\cosh\gamma l + a_l[C\cosh\gamma l + D\sinh\gamma l] = 0. \end{aligned}$$
Therefore, the eigenvalues satisfy
$$\tanh\gamma l = -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l},$$
and the corresponding eigenfunctions are
$$X(x) = C\cosh\gamma x + \frac{a_0}{\gamma} C\sinh\gamma x,$$
where $C$ is a constant. $\square$

4. It is easily known that the rational curve $y = -\frac{(a_0+a_l)\gamma}{\gamma^2+a_0 a_l}$ has a single maximum at $\gamma = \sqrt{a_0 a_l}$ and is monotone in the two intervals $(0, \sqrt{a_0 a_l})$ and $(\sqrt{a_0 a_l}, \infty)$. Furthermore,
$$\max_{\gamma\in[0,\infty)} y(\gamma) = -\frac{a_0 + a_l}{2\sqrt{a_0 a_l}} \geq 1, \ \lim_{\gamma\to\infty} y(\gamma) = 0, \text{ for } y'(0) = -\frac{a_0 + a_l}{a_0 a_l}.$$
Note that $\tanh\gamma l$ is monotone in $[0, \infty)$,
$$\tanh\gamma l < 1 \text{ when } \gamma \in [0, \infty), \ \lim_{\gamma\to\infty} \tanh\gamma l = 1, \text{ and } (\tanh\gamma l)'|_{\gamma=0} = l > -\frac{a_0 + a_l}{a_0 a_l}.$$
Therefore, the rational curve $y = -\frac{(a_0+a_l)\gamma}{\gamma^2+a_0 a_l}$ and the curve $y = \tanh\gamma l$ intersect at two points, that is, there are two negative eigenvalue. $\square$

5. When $\lambda = \beta^2 > 0$, $\beta$ satisfies (10), i.e.
$$\tan\beta l = \frac{(a_0 + a_l)\beta}{\beta^2 - a_0 a_l}.$$
Since $y = \tan\beta l$ is monotonically increasing when $\beta \in ((n - \frac{1}{2})\pi/l, (n + \frac{1}{2})\pi/l)$ $(n = 0, 1, 2, \cdots)$ and
$$\lim_{\beta\to(n-\frac{1}{2})\pi/l} \tan\beta l = -\infty, \quad \lim_{\beta\to(n+\frac{1}{2})\pi/l} \tan\beta l = \infty,$$
while $y = \frac{(a_0+a_l)\beta}{\beta^2-a_0 a_l}$ is negative, monotonically increasing when $\beta \in (\sqrt{a_0 a_l}, \infty)$ and
$$\lim_{\beta\to\infty} \frac{(a_0 + a_l)\beta}{\beta^2 - a_0 a_l} = 0,$$
the two curves intersects at infinite many points, that is, there are an infinite many number of positive eigenvalues. The graph is similiar to the Figure 1 in Section 4.3 in the textbook but $y = \frac{(a_0+a_l)\beta}{\beta^2-a_0 a_l}$ is positive first and then negative now. $\square$

6. (a) If $a > 0$, the case turns out to be case 1 in Section 4.3 and thus there are no negative eigenvalues; if $a = 0$, the case turns out to be the Neumann boundary condition problem and thus there are no negative eigenvalues, either;
if $-2/l \le a < 0$, we have $(\tanh \gamma l)'|_{\gamma=0} = l \le -\dfrac{a_0 + a_l}{a_0 a_l} = -\dfrac{2}{a}$, using the same way as Exercise 4.3.4 above, we conclude that there is only one negative eigenvalue;
if $a < -2l$, we have $(\tanh \gamma l)'|_{\gamma=0} = l > -\dfrac{a_0 + a_l}{a_0 a_l} = -\dfrac{2}{a}$ and thus there are two negative eigenvalues.
   (b) Exercise 4.3.2 implies that $\lambda = 0$ is an eigenvalue if and only if $a_0 + a_l = -a_0 a_l l$, i.e., $a = 0$ or $a = -2l$. ☐

7. Under the condition $a_0 = a_l = a$, the eigenvalue satisfies

$$\lambda = \beta^2,\ \tan \beta l = \frac{2a\beta}{\beta^2 - a^2}.$$

Hence, when $a \to \infty$ and $\frac{n\pi}{l} < \beta_n < \frac{(n+1)\pi}{l}$, $\frac{2a\beta}{\beta^2 - a^2}$ is negative and tends to 0. So Figure 1 in Section 4.3 implies

$$\lim_{a\to\infty} \left\{ \beta_n(a) - \frac{(n+1)\pi}{l} \right\} = 0. \qquad \square$$

9. (a) If $\lambda = 0$, then $X(x) = ax + b$ for some constants $a$ and $b$. Then the boundary conditions imply $a + b = 0$. Therefore, $X_0(x) = ax - a$ for some nonzero constant $a$.
   (b) If $\lambda = \beta^2$, then $X(x) = A\cos\beta x + B\sin\beta x$. Then the boundary conditions imply

$$A + B\beta = 0,\ A\cos\beta + B\sin\beta = 0.$$

   Since $A, B$ can not both be 0, we have $\beta = \tan\beta$.
   (c) omit.
   (d) If $\lambda = -\gamma^2$, then $X(x) = Ae^{\gamma x} + Be^{-\gamma x}$ and

$$A + B + A\gamma - B\gamma = 0,\ Ae^{\gamma} + Be^{-\gamma} = 0.$$

   Then we find the coefficent matrix $\begin{pmatrix} 1+\gamma & 1-\gamma \\ e^{2\gamma} & e^{-\gamma} \end{pmatrix}$ is always nonsingular(since $e^{\gamma} > \frac{1+\gamma}{1-\gamma}$ when $\gamma > 0$, verify by yourself!), then $a = b = 0$. So we conclude that there is not any negative eigenvalue. ☐

10. Let $u(x,t) = X(x)T(t)$, by the summary on Page 97, we can have

$$u(x,t) = \sum_{n=1}^{\infty} (C_n \cos\beta_n ct + D_n \sin\beta_n ct)(\cos\beta_n x + \frac{a_0}{\beta_n}\sin\beta_n x) + (C_0\cosh\gamma ct + D_0 \sinh\gamma ct)(\cosh\gamma x + \frac{a_0}{\gamma}\sinh\gamma x),$$

where $\gamma$ is determined by the intersection point of $\tanh\gamma l = -\frac{(a_0+a_l)\gamma}{\gamma^2 + a_0 a_l}$, and the intitial conditions are

$$\phi(x) = \sum_{n=1}^{\infty} C_n(\cos\beta_n x + \frac{a_0}{\beta_n}\sin\beta_n x) + C_0(\cosh\gamma x + \frac{a_0}{\gamma}\sinh\gamma x),$$
$$\psi(x) = \sum_{n=1}^{\infty} D_n\beta_n c(\cos\beta_n x + \frac{a_0}{\beta_n}\sin\beta_n x) + D_0\gamma c(\cosh\gamma x + \frac{a_0}{\gamma}\sinh\gamma x). \qquad \square$$

11. (a) By the wave equation,

$$\begin{aligned}\frac{dE}{dt} &= \int_0^l [\frac{1}{c^2}u_t u_{tt} + u_x u_{xt}]dx \\ &= \int_0^l [u_t u_{xx} + u_x u_{xt}] \\ &= (u_t u_x)\Big|_0^l = u_t(l,t)u_x(l,t) - u_t(0,t)u_x(0,t).\end{aligned}$$

The Dirichlet boundary conditions $u(l,t) = u(0,t) = 0$ imply $u_t(l,t) = u_x(l,t) = 0$. Hence, $\frac{dE}{dt} \equiv 0$.

(b) Same as above. Omit here.

(c) By the computation in (a) and the Robin boundary conditions, we can get that

$$\frac{dE_R}{dt} = u_t u_x\Big|_0^l + a_l u_t(l,t)u(l,t) + a_0 u_t(0,t)u_x(0,t) \equiv 0. \qquad \square$$

12. (a) Let $\lambda = 0$, we have $v(x) = Ax + B$. Since $v(x) = Ax + B$ sitisfy the boundary conditions for any $A$ and $B$, $\lambda = 0$ is a double eigenvalue.

(b) Let $\lambda = \beta^2 > 0$ and suppose $\beta > 0$, we have $v(x) = C\cos\beta x + D\sin\beta x$. Then boundary conditions imply

$$D\beta = -C\beta\sin\beta l + D\beta\cos\beta l = \frac{C\cos\beta l + D\sin\beta l - C}{l}.$$

Therefore, eigenvalues $\lambda > 0$ satisfies the equation

$$\lambda = \beta^2,\ \sin\beta l(-\sin\beta l + \beta l) = (1 - \cos\beta l)^2.$$

(c) Let $\gamma = \frac{1}{2}l\sqrt{\lambda}$, then $\gamma$ is a root of the following equation

$$\gamma\sin\gamma\cos\gamma = \sin^2\gamma.$$

(d) By (c), we have $\sin\gamma = 0$ or $\gamma = \tan\gamma$. So the positive eigenvalues are $\frac{4n^2\pi^2}{l^2}$ and $4\gamma_n^2/l^2$ where $\gamma_n = \tan\gamma_n \in (n\pi - \pi, n\pi - \frac{\pi}{2})$ for $n = 1, 2, \cdots$. The graph is omitted here.

(e) By (a) and (d), for $\lambda = 0$, the eigenfuntions are 1 and $x$; for $\lambda = \frac{4n^2\pi^2}{l^2}$, $n = 1, 2, \cdots$, the eigenfunctions are $\cos(\frac{2n\pi x}{l})$; for $\lambda = \frac{4\gamma_n^2}{l^2}$, where $\gamma_n = \tan\gamma_n \in (n\pi - \pi, n\pi - \frac{1}{2}\pi)$, $n = 1, 2, \cdots$, the eigenfunctions are

$$\gamma_n\cos\frac{2\gamma_n x}{l} - \sin\frac{2\gamma_n x}{l}.$$

(f) From above, we have

$$\begin{aligned}u(x,t) = A + Bx &+ \sum_{n=1}^{\infty} C_n e^{-\frac{4\gamma_n^2}{l^2}kt}[\gamma_n\cos\frac{2\gamma_n x}{l} - \sin\frac{2\gamma_n x}{l}] \\ &+ \sum_{n=1}^{\infty} D_n e^{-\frac{4n^2\pi^2}{l^2}kt}\cos\frac{2n\pi x}{l}.\end{aligned}$$

(g) By (f), we have $\lim_{t\to\infty} u(x,t) = A + Bx$ since $\lim_{t\to\infty} e^{-\lambda kt} = 0$. $\square$

15. Let $\lambda = \beta^2$, then

$$X(x) = A\cos\frac{\beta\rho_1 x}{\kappa_1} + B\sin\frac{\beta\rho_1 x}{\kappa_1},\ 0 < x < a;$$
$$X(x) = C\cos\frac{\beta\rho_2 x}{\kappa_2} + D\sin\frac{\beta\rho_2 x}{\kappa_2},\ a < x < l.$$

Hence, the boundary condtions imply

$$A = 0; \qquad C\cos\frac{\beta\rho_2 l}{\kappa_2} + D\sin\frac{\beta\rho_2 l}{\kappa_2} = 0;$$
$$A\cos\frac{\beta\rho_1 a}{\kappa_1} + B\sin\frac{\beta\rho_1 a}{\kappa_1} = C\cos\frac{\beta\rho_2 a}{\kappa_2} + D\sin\frac{\beta\rho_2 a}{\kappa_2};$$
$$-A\frac{\beta\rho_1}{\kappa_1}\sin\frac{\beta\rho_1 a}{\kappa_1} + B\frac{\beta\rho_1}{\kappa_1}\cos\frac{\beta\rho_1 a}{\kappa_1} = -C\frac{\beta\rho_2}{\kappa_2}\sin\frac{\beta\rho_2 a}{\kappa_2} + D\frac{\beta\rho_1}{\kappa_1}\cos\frac{\beta\rho_2 a}{\kappa_2}.$$

Hence, when the eigenvalue is positive, i.e. $\lambda = \beta^2 > 0$, $\beta$ satisfies

$$\frac{\rho_1}{\kappa_1}\cot\frac{\beta\rho_1 a}{\kappa_1} + \frac{\rho_2}{\kappa_2}\cot\frac{\beta\rho_2(l-a)}{\kappa_2} = 0.$$

Let $\lambda = 0$, then the boundary conditions imply

$$X(x) = \begin{cases} Ax & 0 < a < l; \\ B(x-l) & a < x < l. \end{cases}$$

Since $X(x)$ should be differentiable at $x = a$, such $A$ and $B$ can not exist except $A = B = 0$.
Let $\lambda = -\gamma^2 < 0$, then

$$X(x) = A\cosh\frac{\beta\rho_1 x}{\kappa_1} + B\sinh\frac{\beta\rho_1 x}{\kappa_1},\ 0 < x < a;$$
$$X(x) = C\cosh\frac{\beta\rho_2 x}{\kappa_2} + D\sinh\frac{\beta\rho_2 x}{\kappa_2},\ a < x < l.$$

Hence, the boundary condtions imply

$$A = 0; \qquad C\cosh\frac{\beta\rho_2 l}{\kappa_2} + D\sinh\frac{\beta\rho_2 l}{\kappa_2} = 0;$$
$$A\cosh\frac{\beta\rho_1 a}{\kappa_1} + B\sinh\frac{\beta\rho_1 a}{\kappa_1} = C\cosh\frac{\beta\rho_2 a}{\kappa_2} + D\sinh\frac{\beta\rho_2 a}{\kappa_2};$$
$$A\frac{\beta\rho_1}{\kappa_1}\sinh\frac{\beta\rho_1 a}{\kappa_1} + B\frac{\beta\rho_1}{\kappa_1}\cosh\frac{\beta\rho_1 a}{\kappa_1} = C\frac{\beta\rho_2}{\kappa_2}\sinh\frac{\beta\rho_2 a}{\kappa_2} + D\frac{\beta\rho_1}{\kappa_1}\cosh\frac{\beta\rho_2 a}{\kappa_2}.$$

Hence, when the eigenvalue is negative, i.e. $\lambda = \beta^2 > 0$, $\beta$ satisfies

$$\frac{\rho_1}{\kappa_1}\coth\frac{\beta\rho_1 a}{\kappa_1} + \frac{\rho_2}{\kappa_2}\coth\frac{\beta\rho_2(l-a)}{\kappa_2} = 0.$$

However, since the left handside is always positive. Therefore, there is no negative eigenvalues. □

16. Let $\lambda = \beta^4 > 0$ where $\beta > 0$, and $X(x) = A\cosh\beta x + B\sinh\beta x + C\cos\beta x + D\sin\beta x$. By the boundary conditions
$$\beta_n = \frac{n\pi}{l},\ \lambda_n = (\frac{n\pi}{l})^4,\ X_n(x) = \sin\frac{n\pi x}{l},\ n = 1, 2, \cdots.$$
The details are as the following exercise. □

17. Let $\lambda = \beta^4 > 0$ where $\beta > 0$, and $X(x) = A\cosh\beta x + B\sinh\beta x + C\cos\beta x + D\sin\beta x$. Hence by the boundary conditions,
$$\begin{aligned} A + C &= 0,\\ B + D &= 0,\\ A\cosh\beta l + B\sinh\beta l + C\cos\beta l + D\sin\beta l &= 0,\\ A\sinh\beta l + B\cosh\beta l - C\sin\beta l + D\cos\beta l &= 0, \end{aligned}$$
which simplifies to
$$A(\cosh\beta l - \cos\beta l) + B(\sinh\beta l - \sin\beta l) = 0, A(\sinh\beta l + \sin\beta l) + B(\cosh\beta l - \cos\beta l) = 0.$$
Since eigenfunctions are nontrivial, the determinant of the matrix should be zero, that is,
$$(\cosh\beta l - \cos\beta l)^2 - (\sinh^2\beta l - \sin^2\beta l) = 0,$$
$$\cosh\beta l\cos\beta l = 1$$
and the corresponding eigenfunction is
$$X(x) = (\sinh\beta l - \sin\beta l)(\cosh\beta x - \cos\beta x) - (\cosh\beta l - \cos\beta l)(\sinh\beta x - \sin\beta x).$$ □

Problem 10. $u(x,t) = X(x)T(t) \Longrightarrow -\frac{T''(t)}{a^2T(t)} = \frac{X^{(4)}(x)}{X(x)} = \lambda \Longrightarrow X^{(4)} - \lambda X = 0$ and $T'' + \lambda a^2 T = 0$

$\Longrightarrow \lambda\int_0^l |X|^2 = \int_0^l X^{(4)}\overline{X} = \int_0^l |X''|^2 \Longrightarrow \lambda = \frac{\int_0^l |X''|^2}{\int_0^l |X|^2} \geq 0$

If $\lambda = 0$, then$X'' \equiv 0 \Longrightarrow X(x) = ax + b \Longrightarrow X \equiv 0$ since $X(0) = X(l) = 0 \Longrightarrow \lambda > 0$

Let$\lambda = \beta^4, \beta > 0$, then
$$T(t) = A\cos(\beta^2 at) + B\sin(\beta^2 at)$$
$$X(x) = Ce^{\beta x} + De^{-\beta x} + E\cos(\beta x) + F\sin(\beta x)$$
$u(0,t) = u_{xx}(0,t) = u(l,t) = u_{xx}(l,t) = 0 \Longrightarrow X(0) = X(l) = X''(0) = X''(l) = 0 \Longrightarrow$

$E = 0, F\sin(\beta l) = 0, C = -D = 0 \Longrightarrow \sin(\beta l) = 0 \Longrightarrow$

$\beta_n = \frac{n\pi}{l}, X_n(x) = \sin(\beta_n l), (n = 1, 2, 3, ...)$ are distinct solutions.

$\Longrightarrow u(x,t) = \sum_{n=1}^{\infty}(A_n\cos(\beta_n^2 at) + B_n\sin(\beta_n^2 at))\sin(\beta_n l)$ where $A_n$, $B_n$are determined by
$$\phi(x) = u(x,0) = \sum_{n=1}^{\infty} A_n\sin(\beta_n l)$$
$$\psi(x) = u_t(x,0) = \sum_{n=1}^{\infty} \beta_n^2 aB_n\sin(\beta_n l)$$