Midterm Exam

Introductory Probability

Calculators, books, notes and extra papers are not allowed on this exam.
Please show all your work and explain all answers to qualify for FULL credit.

1. (a) (10 points) Independent trials consisting of rolling two dice are performed. What is the probability that an outcome of 6 appears before an outcome of 9, when the outcome of a roll is the sum of the two dice?

   (b) (10 points) Generalize the conclusion. (Hint: Instead of "an outcome of 6" and "an outcome of 9", in general one may consider two events of an experiment and perform independent trials of this experiment and then, consider which event appears earlier.)

2. Consider an ordinary deck of 52 playing cards. (For this problem, there is NO need to simplify your answers. In other words, the solutions can be expressed with number of combinations.)

   (a) (10 points) A hand of 5 cards is said to be a Three of a Kind, if there are three distinct denominations in the hand; one denomination occurs three times and the other two denominations occur once each. Similarly, a hand of 5 cards is said to be a Four of a Kind, if there are two distinct denominations in the hand; one denomination occurs four times and the other occurs once.

   What is the probability that one is dealt a Three of a Kind? What is the probability that one is dealt a Four of a Kind?

   (b) (10 points) What is the probability that one is dealt a hand of 13 cards which contains all 4 of at least 1 of the 13 denominations? (Hint: One may apply inclusive-exclusive identity.)

3. (a) (10 points) Let P be a probability function on a sample space S. Let E, F be two events such that $P(E), P(F) \in (0, 1)$. Prove the following Bayes formula:

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^{\mathrm{C}})P(F^{\mathrm{C}})},$$

   where F^{C} stands for the complement of the event F.

   (b) (10 points) Consider two boxes. The first one contains 1 black and 1 white marble, and the other one contains 2 black and 1 white marble. A box is selected at random, and then a marble is drawn from the selected box at random. What is the probability that the first box was the one selected, given that the chosen marble is white?

4. An urn initially contains one red and one blue ball. At each stage, a ball is randomly chosen and then replaced ALONG with another ball of the same color. Let X denote the selection number of the first chosen ball that is blue. For instance, at the first stage we pick out a red ball, then we put back two red balls such that the urn now contains three balls; if the second selection is blue, then the experiment stops and X is equal to 2.

(a) (10 points) Find the probability mass function of X.

(b) (5 points) Prove that $P(X < \infty) = 1$. This means, with probability 1, a blue ball is eventually chosen.

(c) (5 points) Find the expectation of X.

5. Let X be a Poisson random variable with parameter $\lambda > 0$.

(a) (5 points) Write down the probability mass function of X, as well as the the expectation of X. (For the expectation, no proof is needed.)

(b) (10 points) Let $n \geq 1$ be an integer. We call $E[X^n]$, the expectation of X^n, the nth moment of X. Prove the following recursive formula of moments of X:

$$E[X^{n+1}] = \lambda\, E[(X+1)^n], \quad \text{when } n \geq 1.$$

(c) (5 points) Find the third moment of X.