Final Exam

Introductory Probability

Calculators, books, notes and extra papers are not allowed on this exam.
Please show all your work and explain all answers to qualify for FULL credit.

1. We consider in this problem a one dimensional random walk. Let n be a positive integer. Let $X_1, \cdots, X_n$ be independent and identically distributed discrete random variables, each having the following probability mass function

$$P(X_i = -1) = P(X_i = 1) = 1/2, \; i = 1, \cdots, n.$$

Denote $X = X_1 + \cdots + X_n$. Find

   (a) (10 points) $E[X]$.

   (b) (10 points) $E[X^2]$.

2. Let X be an exponential random variable with parameter $\lambda > 0$.

   (a) (10 points) Prove that X is memoryless, in the sense that, for any positive s and t,

$$P(X > s + t \,|\, X > t) = P(X > s).$$

   (b) (10 points) Find the probability density function of $\log X$.

3. We show in this problem that the sum of two independent normal random variables is still a normal one.

   (a) (5 points) Let Z be a standard normal random variable. Find the moment generating function of Z.

   (b) (5 points) Let X be a normal random variable with parameters (μ, σ^2), where $\sigma > 0$. Find the moment generating function of X.

   (c) (10 points) Now let X_1, X_2 be independent normal random variables with parameters (μ_1, σ_1^2) and (μ_2, σ_2^2), respectively, where $\sigma_1, \sigma_2 > 0$. Denote $X = X_1 + X_2$. Prove that X is a normal random variable. Also, find the corresponding parameters.

4. Let $(X_n)_{n \geq 0}$ be a Markov chain with state space $I = \{0, 1, 2\}$ and transition matrix

$$P = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 3/4 & 0 & 1/4 \\ 0 & 1 & 0 \end{pmatrix}.$$

   (a) (5 points) Assume that the initial distribution of the system is such that $P(X_0 = 0) = P(X_0 = 1) = P(X_0 = 2) = 1/3$. Find the probability mass function of X_2.

(b) (5 points) Prove that the Markov chain is ergodic, in the sense that there exists some positive integer n such that all entries of the matrix P^n are positive.

(c) (5 points) Find the stationary distribution of the system.

(d) (5 points) Find $\lim_{n\to\infty} P^n$.

5. Alice and Bob have agreed to meet for lunch between 12 noon and 1 p.m. Let X and Y denote, respectively, the time past 12 noon (in hour) that Alice and Bob arrive. Assume that the joint probability density function of X and Y is given by

$$f(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & 0 \le x \le 1,\, 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) (5 points) Find the marginal density functions of X and Y.

(b) (5 points) Are X and Y independent? Why?

(c) (5 points) Find the probability that Alice arrives later than Bob.

(d) (5 points) Find the expected amount of time (in hour) that the one who arrives first must wait for the other person.