Solution to Midterm

1. (a) Let E_1, E_2 be the events that the outcome is 6 and 9 respectively. Let E be the event that E_1 occurs first. By direct counting,

$$P(E_1) = \frac{5}{36}, \quad P(E_2) = \frac{4}{36}.$$

Method 1:

$$\begin{aligned} P(E) &= P(E|E_1)P(E_1) + P(E|E_2)P(E_2) + P(E|(E_1 \cup E_2)^c)P((E_1 \cup E_2)^c) \\ &= P(E_1) + P(E)P((E_1 \cup E_2)^c) \qquad \text{(independence)} \\ &= \frac{P(E_1)}{P(E_1 \cup E_2)} \end{aligned}$$

Method 2:

$$\begin{aligned} P(E) &= \sum_{i=0}^{\infty} (1 - P(E_1) - P(E_2))^i P(E_1) \\ &= \frac{P(E_1)}{P(E_1) + P(E_2)} \end{aligned}$$

(b) Statement: Let E_1 and E_2 be 2 mutually exclusive events of a trial which is to be repeated indefinitely. Then the probability of E_2 occurs before E_1 is $\frac{P(E_2)}{P(E_1 \cup E_2)}$. The proof is identical to that part (a).

2. (a)

$$P(\text{three of a kind}) = \frac{C_3^{13}C_1^3C_3^4C_1^4C_1^4}{C_5^{52}}, \quad P(\text{four of a kind}) = \frac{C_2^{13}C_1^2C_4^4C_1^4}{C_5^{52}}$$

(b) Let E_i be the event that all 4 of the number i cards are in the hand. For distinct i, j, k,

$$P(E_i) = \frac{C_9^{48}}{C_{13}^{52}}, \quad P(E_iE_j) = \frac{C_5^{44}}{C_{13}^{52}}, \quad P(E_iE_jE_k) = \frac{C_1^{40}}{C_{13}^{52}}.$$

Note also any 4 or more intersections of these events has probability 0. By the inclusion-exclusion principle, the required probability is

$$\begin{aligned} P\left(\bigcup_{i=1}^{13} E_i\right) &= \sum_{i=1}^{13} P(E_i) - \sum_{i<j} P(E_iE_j) + \sum_{i<j<k} P(E_iE_jE_k) \\ &= C_1^{13}\frac{C_9^{48}}{C_{13}^{52}} - C_2^{13}\frac{C_5^{44}}{C_{13}^{52}} + C_3^{13}\frac{C_1^{40}}{C_{13}^{52}}. \end{aligned}$$

3. (a)

$$\begin{aligned} P(E|F) &= \frac{P(EF)}{P(F)} \\ &= \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F|E^c)P(E^c)} \end{aligned}$$

(b) Let E be the event that the first box is chosen and F be the event that the chosen marble is white. By Bayes formula,

$$\begin{aligned} P(E|F) &= \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F|E^c)P(E^c)} \\ &= \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{2}} \\ &= \frac{3}{5} \end{aligned}$$

4. (a)

$$P(X = n) = \frac{1}{2}\frac{2}{3}\cdots\frac{n-1}{n}\frac{1}{n+1} = \frac{1}{n(n+1)}$$

(b)

$$P(X < \infty) = 1 - P(X = \infty) = 1 - \lim_{n\to\infty}\frac{1}{2}\frac{2}{3}\cdots\frac{n}{n+1} = 1 - \lim_{n\to\infty}\frac{1}{n+1} = 1$$

(c)

$$\begin{aligned} E[X] &= \sum_{k=1}^{\infty} kP(X = k) \\ &= \sum_{k=1}^{\infty} k\frac{1}{k(k+1)} \\ &= \sum_{k=2}^{\infty}\frac{1}{k} = \infty \end{aligned}$$

5. (a)

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, k = 0, 1, 2, \cdots, \quad E[X] = \lambda$$

(b)

$$\begin{aligned} E[X^{n+1}] &= \sum_{k=0}^{\infty} k^{n+1}\frac{\lambda^k}{k!} \\ &= \lambda\sum_{k=1}^{\infty} k^n\frac{\lambda^{k-1}}{(k-1)!} \\ &= \lambda\sum_{k=0}^{\infty}(k+1)^n\frac{\lambda^k}{k!} \\ &= \lambda E[(X+1)^n] \end{aligned}$$

(c) By (b),

$$E[X^3] = \lambda E[(X+1)^2] = \lambda E[X^2+2X+1] = \lambda(\mathrm{Var}(X)+E[X]^2+2E[X]+1) = \lambda(\lambda^2+3\lambda+1).$$