**Solution to Final**

1. (a)
$$E[X_i] = (1)P(X_i = 1) + (-1)P(X_i = -1) = 0$$
$$E[X] = \sum_{i=1}^{n} E[X_i] = 0$$

   (b) Since the $X_i$'s are independent,
$$E[X^2] = \sum_{i=1}^{n} E[X_i^2] + 2\sum_{i<j} E[X_i]E[X_j] = n(1) + 2C_2^n(0) = n.$$

2. (a) Recall that $P(X > t) = e^{-\lambda t}$. Then
$$P(X > s+t | X > t) = \frac{P(X > s+t)}{P(X > t)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = e^{-\lambda s} = P(X > s).$$

   (b) Let $Y = \log X$. Then for $t \in \mathbb{R}$
$$\begin{aligned} P(Y \le t) &= P(X \le e^t) \\ F_Y(t) &= F_X(e^t) \\ f_Y(t) &= e^t f_X(e^t) = \lambda \exp(t - \lambda e^t) \end{aligned}$$

3. (a)
$$\begin{aligned} M_Z(t) &= E[e^{tZ}] \\ &= \int_{-\infty}^{\infty} e^t \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\ dx \\ &= e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+t)^2}{2}}\ dx \\ &= e^{\frac{t^2}{2}} \end{aligned}$$

   (b)
$$M_X(t) = E[e^{tX}] = E[e^{\mu t} e^{(\sigma t)Z}] = e^{\mu t + \frac{(\sigma t)^2}{2}}$$

   (c) Since $X_1, X_2$ are independent,
$$M_X(t) = M_{X_1}(t) M_{X_2}(t) = \exp\left((\mu_1 + \mu_2)t + \frac{(\sigma_1^2 + \sigma_2^2)t^2}{2}\right).$$

   By comparing it with the moment generating function of s normal random variable with parameters $(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ and recalling that moment generating functions are 1-1 corresponding to the distributions, we get
$$X \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2).$$

4. (a) We compute
$$P^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \\ \frac{3}{4} & 0 & \frac{1}{4} \end{pmatrix}.$$

   Hence, the required probabilities are
$$\begin{pmatrix} P(X_2 = 0) & P(X_2 = 1) & P(X_2 = 2) \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} P^2 = \begin{pmatrix} \frac{13}{24} & \frac{5}{18} & \frac{13}{72} \end{pmatrix}.$$

(b) We compute

$$P^3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{9}{16} & \frac{1}{4} & \frac{3}{16} \\ \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \end{pmatrix}.$$

Hence, the chain is ergodic.

(c) Since the chain is ergodic, the unique stationary probability $\pi$ is given by the solution to the linear system

$$\begin{cases} \pi P & = \pi \\ \pi_1 + \pi_2 + \pi_3 & = 1 \end{cases}.$$

On solving,

$$\pi = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}.$$

(d) Recall that we have

$$\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$$

and hence

$$\lim_{n\to\infty} P^n = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}.$$

5. (a)

$$f_X(x) = \int_0^1 \frac{6}{5}(x+y^2)dy = \frac{6}{5}\left(x + \frac{1}{3}\right), \quad 0 \le x \le 1$$

$$f_Y(y) = \int_0^1 \frac{6}{5}(x+y^2)dx = \frac{6}{5}\left(\frac{1}{2} + y^2\right), \quad 0 \le y \le 1$$

(b) Since

$$f(x,y) \ne f_X(x)f_Y(y) \quad \text{for some } 0 \le x, y \le 1,$$

$X$ and $Y$ are dependent.

(c)

$$\begin{aligned} P(X > Y) &= \iint_{X>Y} f(x,y)\ dxdy \\ &= \int_0^1 \int_0^x \frac{6}{5}(x+y^2)dydx \\ &= \int_0^1 \frac{6}{5}\left(x^2 + \frac{1}{3}x^3\right) dx \\ &= \frac{1}{2} \end{aligned}$$

(d)

$$\begin{aligned} E[|X-Y|] &= \int_0^1 \int_0^1 |x-y| f(x,y) dxdy \\ &= \int_0^1 \int_0^x (x-y)\frac{6}{5}(x+y^2)dydx + \int_0^1 \int_0^y (y-x)\frac{6}{5}(x+y^2)dxdy \\ &= \int_0^1 \frac{6}{5}\left(\frac{1}{2}x^3 + \frac{1}{12}x^4\right) dx + \int_0^1 \frac{6}{5}\left(\frac{1}{6}y^3 + \frac{1}{2}y^4\right) dy \\ &= \frac{17}{50} \end{aligned}$$