Solution to Assignment 5

1. (a) Note that the function is always positive and thus the order of integration is irrelevant. It is clearly continuous and non-negative and thus we only need to check that

$$\int_0^1 \int_0^2 f(x,y)\ dydx = 1.$$

Now,

$$\int_0^1 \int_0^2 f(x,y)\ dydx = \int_0^1 \int_0^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)\ dydx = \int_0^1 \frac{6}{7}(2x^2+x)\ dx = 1.$$

(b)

$$f_X(x) = \int_0^2 f(x,y)\ dy = \frac{6}{7}\left(2x^2+x\right)$$

(c)

$$\begin{aligned} P\{X > Y\} &= \int_0^1 \int_0^x f(x,y)\ dydx \\ &= \int_0^1 \frac{6}{7}\left(x^3 + \frac{x^3}{4}\right) dx \\ &= \frac{6}{7}\frac{5}{4}\frac{1}{4} = \frac{15}{56} \end{aligned}$$

(d)

$$P\left\{X < \frac{1}{2}\right\} = \int_0^{\frac{1}{2}} \frac{6}{7}\left(2x^2+x\right)\ dx = \frac{6}{7}\left(\frac{2}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{2}\left(\frac{1}{2}\right)^2\right) = \frac{5}{56}$$

$$P\left\{X, Y < \frac{1}{2}\right\} = \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)\ dydx = \int_0^{\frac{1}{2}} \frac{3}{7}\left(x^2 + \frac{x}{4}\right)\ dx = \frac{1}{32}$$

$$P\left\{Y < \frac{1}{2} \middle| X < \frac{1}{2}\right\} = \frac{7}{20}$$

(e)

$$E[X] = \int_0^1 \frac{6}{7}x(2x^2+x)\ dx = \frac{5}{7}$$

(f)

$$f_Y(y) = \int_0^1 f(x,y)\ dx = \frac{6}{7}\left(\frac{1}{3} + \frac{y}{4}\right) = \frac{1}{14}\left(4+3y\right)$$

$$E[Y] = \int_0^2 \frac{1}{14}y(4+3y)\ dy = \frac{8}{7}$$

2. (a) Note that

$$f(x,y) = 12 \cdot [x(1-x)] \cdot y, \quad 0 < x < 1, 0 < y < 1$$

and the region is the product of intervals, so X and Y are independent.

(b)

$$E[X] = \int_0^1 \int_0^1 xf(x,y)\ dydx = \int_0^1 6x^2(1-x)dx = \frac{1}{2}$$

(c)

$$E[Y] = \int_0^1 \int_0^1 yf(x,y)\ dxdy = \int_0^1 2y^2 dx = \frac{2}{3}$$

(d)

$$\begin{aligned} F_{X^2}(t) &= F_X(\sqrt{t}) \\ f_{X^2}(t) &= \frac{1}{2\sqrt{t}} f_X(\sqrt{t}) \\ &= \frac{1}{2\sqrt{t}} 6\sqrt{t}(1-\sqrt{t}) \\ &= 3(1-\sqrt{t}) \end{aligned}$$

$$\begin{aligned} E[X^2] &= \int_0^1 3x(1-\sqrt{x})dx = \frac{3}{2} \\ \text{Var}(X) &= E[X^2] - (E[X])^2 = \frac{5}{4} \end{aligned}$$

(e) Similarly,

$$\begin{aligned} f_{Y^2}(y) &= \frac{1}{2\sqrt{t}} f_Y(\sqrt{t}) = 1 \\ \text{Var}(Y) &= E[Y^2] - (E[Y])^2 = \frac{1}{18} \end{aligned}$$

3. (a) Let $X_i = 1$ if the i-th man has a woman next to him and 0 otherwise. Let E be the event the man is lined up at the beginning or the end of the line.

$$\begin{aligned} P(X_i = 1) &= P(X_i = 1|E)P(E) + P(X_i = 1|E^c)P(E^c) \\ &= \frac{2}{2n}\frac{n}{2n-1} + \frac{2n-2}{2n}\left(\frac{3}{2}\frac{n}{2n-1}\right) \\ &= \frac{3n-1}{2(2n-1)} \end{aligned}$$

Hence,

$$E\left(\sum_{i=1}^{n} X_i\right) = \frac{n(3n-1)}{2(2n-1)}$$

(b) Now,

$$P(X_i = 1) = \frac{3}{2}\frac{n}{2n-1}.$$

Thus,

$$E\left(\sum_{i=1}^{n} X_i\right) = \frac{3n^2}{2(2n-1)}$$

4.

$$
\begin{aligned}
E[X] &= \int_0^\infty \int +0^\infty x f(x,y)\ dxdy \\
&= \int_0^\infty \frac{e^{-y}}{y} \left(\int_0^\infty x e^{-\frac{x}{y}}\ dx \right) dy \\
&= \int_0^\infty \frac{e^{-y}}{y} \left(-xye^{-\frac{x}{y}} \Big|_0^\infty + y \int_0^\infty e^{-\frac{x}{y}}\ dx \right) dy \\
&= \int_0^\infty \frac{e^{-y}}{y} \left(-y^2 e^{-\frac{x}{y}} \right) \Big|_0^\infty\ dy \\
&= \int_0^\infty y e^{-y}\ dy \\
&= -ye^{-y} \big|_0^\infty + \int_0^\infty e^{-y}\ dy \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
E[Y] &= \int_0^\infty \int_0^\infty y f(x,y)\ dxdy \\
&= \int_0^\infty e^{-y} \left(\int_0^\infty e^{-\frac{x}{y}}\ dx \right) dy \\
&= \int_0^\infty y e^{-y}\ dy \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
E[XY] &= \int_0^\infty \int_0^\infty xy f(x,y)\ dxdy \\
&= \int_0^\infty e^{-y} \left(\int_0^\infty x e^{-\frac{x}{y}}\ dx \right) dy \\
&= \int_0^\infty y^2 e^{-y}\ dy \\
&= -y^2 e^{-y} \big|_0^\infty + 2 \int_0^\infty y e^{-y}\ dy \\
&= 2
\end{aligned}
$$

Hence,

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y] = 1.$$

5. Note that

$$M_{\ln X}(t) = E[e^{t \ln X}] = E[X^t]$$

and

$$M_{\ln X}(t) = M_Z(t) = e^{\frac{t^2}{2}}.$$

Hence, $E[X] = e^{\frac{1}{2}}$, $E[X^2] = e^2$ and $\mathrm{Var}(X) = e^2 - e$.

6. (a) Let $X_i = 1$ if the i-th triple consists of players of all types and $X_i = 0$ otherwise.

$$E[X_i] = E[X^2] = P(X_i = 1) = \frac{C_1^2 C_1^3 C_1^4}{C_3^9} = \frac{2}{7}.$$

Hence,

$$E\left(\sum_{i=1}^{3} X_i\right) = \frac{6}{7}.$$

(b)

$$\mathrm{Var}(X_i) = \frac{2}{7} - \left(\frac{2}{7}\right)^2 = \frac{10}{49}$$

and for $i \neq j$,

$$\begin{aligned}
E[X_i X_j] &= P(X_i, X_j = 1) \\
&= P(X_i = 1 | X_j = 1) P(X_j = 1) \\
&= \frac{C_1^1 C_1^2 C_1^3}{C_3^6} \frac{C_1^2 C_1^3 C_1^4}{C_3^9} \\
&= \frac{6}{70}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathrm{Var}\left(\sum_{i=1}^{3} X_i\right) &= \sum_{i=1}^{3} \mathrm{Var}(X_i) + 2\sum_{i<j} \mathrm{Cov}(X_i, X_j) \\
&= 3\frac{10}{49} + 2C_2^3 \left(\frac{6}{70} - \left(\frac{2}{7}\right)^2\right) \\
&= \frac{312}{490}
\end{aligned}$$