Solution to Assignment 4

1. (a)

$$P\{X > 20\} = \int_{20}^{\infty} f(x)\ dx = \int_{20}^{\infty} \frac{10}{x^2}\ dx = \left.\frac{-10}{x}\right|_{20}^{\infty} = \frac{1}{2}$$

(b) For $t < 10$, $F(t) = 0$. For $t \geq 10$,

$$F(t) = \int_{-\infty}^{t} f(x)\ dx = \int_{10}^{t} \frac{10}{x^2}\ dx = \left.\frac{-10}{x}\right|_{10}^{t} = 1 - \frac{10}{t}$$

(c) We assume the life times of the devices are independent. Now,

$$P\{X \geq 15\} = 1 - \frac{10}{15} = \frac{1}{3}.$$

So the required probability is

$$\sum_{k=3}^{6} C_k^6 \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{6-k}$$

2. (a)

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f(x)\ dx \\ &= \int_{0}^{\infty} \frac{1}{4} x^2 e^{-\frac{x}{2}}\ dx \\ &= -\left.\frac{1}{2} x^2 e^{-\frac{x}{2}}\right|_0^{\infty} + \int_0^{\infty} x e^{-\frac{x}{2}}\ dx \\ &= \left.-2x e^{-\frac{x}{2}}\right|_0^{\infty} + 2\int_0^{\infty} e^{-\frac{x}{2}}\ dx \\ &= 4 \end{aligned}$$

(b)

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f(x)\ dx \\ &= \int_{-1}^{1} cx(1 - x^2)\ dx \\ &= \left. c\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\right|_{-1}^{1} \\ &= 0 \end{aligned}$$

(c)

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f(x)\ dx \\ &= \int_{5}^{\infty} \frac{5}{x}\ dx \\ &= 5 \ln x|_5^{\infty} \\ &= \infty \end{aligned}$$

3. Let X be uniformly distributed over $(0,1)$. For $a \in (0,1)$, let $E_a = \{X \neq a\}$. Then $P(E_a) = 1 \ \forall a \in (0,1)$ and

$$P\left(\bigcap_{a\in(0,1)} E_a\right) = P(\{X \neq a \ \forall a \in (0,1)\}) = P(\phi) = 0.$$

4. (a)

$$P\{Z > x\} = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,dt = -\int_{-x}^{-\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(-t)^2}{2}}\,d(-t) = \int_{-\infty}^{-x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,dt = P\{Z < -x\}$$

(b)

$$P\{|Z| > x\} = P\{Z > x\} + P\{Z < -x\} = 2P\{Z > x\}$$

(c)

$$P\{|Z| < x\} = 1 - P\{|Z| > x\} = 1 - 2P\{Z > x\} = 1 - 2(1 - P\{Z < x\}) = 2P\{Z < x\} - 1$$

5. Let $Y = cX$. For $t \geq 0$,

$$F_Y(t) = F_X\left(\frac{t}{c}\right) = 1 - e^{-\lambda\left(\frac{t}{c}\right)} = 1 - e^{\left(\frac{\lambda}{c}\right)t}.$$

Hence, Y is the exponential random variables with parameter $\frac{\lambda}{c}$.