Solution to Assignment 2

1.
$$p = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)P(E_4|E_1E_2E_3) = \frac{C_1^4C_{12}^{48}}{C_{13}^{52}}\frac{C_1^3C_{12}^{36}}{C_{13}^{39}}\frac{C_1^2C_{12}^{24}}{C_{13}^{26}}\frac{C_1^1C_{12}^{12}}{C_{13}^{13}}$$

2. Let R_1, R_2, R_3 be the event that she receives a weak, moderate and strong recommendation respectively and let O be the event that she receives an offer.

   (a)
   $$P(O) = P(O|R_1)P(R_1)+P(O|R_2)P(R_2)+P(O|R_3)P(R_3) = (0.1)(0.1)+(0.4)(0.2)+(0.8)(0.7)$$

   (b)
   $$P(R_1|O) = \frac{P(O|R_1)P(R_1)}{P(O|R_1)P(R_1)+P(O|R_2)P(R_2)+P(O|R_3)P(R_3)}$$
   $$P(R_2|O) = \frac{P(O|R_2)P(R_2)}{P(O|R_1)P(R_1)+P(O|R_2)P(R_2)+P(O|R_3)P(R_3)}$$
   $$P(R_3|O) = \frac{P(O|R_3)P(R_3)}{P(O|R_1)P(R_1)+P(O|R_2)P(R_2)+P(O|R_3)P(R_3)}$$

   (c)
   $$P(R_1|O^c) = \frac{P(O^cR_1)}{P(O^c)} = \frac{P(R_1)-P(OR_1)}{1-P(O)} = \frac{P(R_1)(1-P(O|R_1))}{1-P(O)}$$
   Similarly for the other 2.

3. Let A_i be the event that player A wins the ith game.

   (a) The required probability is
   $$P(A_1A_2^cA_3A_4 + A_1^cA_2A_3A_4 + A_1^cA_2A_3^cA_4^c + A_1A_2^cA_3^cA_4^c) = 2p^3(1-p) + 2p(1-p)^3$$

   (b) Let A be the event that player A wins the game. Note that player A can only win the game when even number of games are played and player A must win a game and then lose a game or lose a game and then win a game for the game to continue (and hence the factor $2p(1-p) = p(1-p)+(1-p)p$ in the following calculation). The required probability is
   $$\begin{aligned} P(A) &= p^2 + 2p(1-p)p^2 + (2p(1-p)^2p^2 + \cdots \\ &= p^2\sum_{i=0}^{\infty}(2p(1-p))^i \\ &= \frac{p^2}{1-2p(1-p)}. \end{aligned}$$

4.
$$P\left(\bigcup_{i=1}^n E_i\right) = 1 - P\left(\bigcap_{i=1}^n E_i^c\right) = 1 - \prod_{i=1}^n P(E_i^c) = 1 - \prod_{i=1}^n (1-P(E_i))$$

5. The trials are not independent.

6.

$$\begin{aligned} P(E|E \cup F) &= \frac{P(E)}{P(E \cup F)} \\ &= \frac{P(E|F)P(F) + P(EF^c)}{P(E \cup F)} \\ &= P(E|F) + \frac{P(E|F)(P(EF) - P(E)) + P(EF^c)}{P(E \cup F)} \\ &= P(E|F) + \frac{P(EF^c)(1 - P(E|F))}{P(E \cup F)} \\ &\geq P(E|F) \end{aligned}$$