Solution to Assignment 1

1. Let R, N be the events that the student picked is wearing a ring and a necklace respectively. It is given that
$$P((R \cup N)^c) = 0.6, \quad P(R) = 0.2, \quad P(N) = 0.3.$$
Hence, the probability that the student is wearing a ring or necklace is
$$P(R \cup N) = 1 - P((R \cup N)^c) = 0.4$$
and the the probability that the student is wearing a ring and necklace is
$$P(R \cap N) = P(R) + P(N) - P(R \cup N) = 0.1$$

2. Without replacement
$$P(\text{same colour}) = P(\text{all red}) + P(\text{all blue}) + P(\text{all green}) = \frac{C_3^5 + C_3^6 + C_3^8}{C^19_3}$$
$$P(\text{all different colour}) = \frac{C_3^5 C_3^6 C_3^8}{C_3^{19}}$$
With replacement
$$P(\text{same colour}) = P(\text{all red}) + P(\text{all blue}) + P(\text{all green}) = \frac{5^3 + 6^3 + 8^3}{19^3}$$
$$P(\text{all different colour}) = \frac{3!(5 \cdot 6 \cdot 8)}{19^3}$$

3. Define
$$F_1 = E_1, \quad F_i = E_i \setminus \left(\cup_{j=1}^{i-1} E_j\right), \ i \geq 2.$$
Claim: $F_i \cap F_j = \phi$ for $i < j$.
Proof:
$$F_i \cap F_j \subseteq E_i \cap (E_j \setminus E_i) = \phi$$
Claim: $\cup_{i=1}^n F_i = \cup_{i=1}^n E_i$ for any $n \geq 1$.
Proof:
$$\cup_{i=1}^n F_i = E_1 \cup (E_2 \setminus E_1) \cup_{i=3}^n F_i = (E_1 \cup E_2) \cup (E_3 \setminus (E_1 \cup E_2)) \cup_{i=4}^n F_i = \cup_{i=1}^3 E_i \cup_{i=4}^n F_i = \cup_{i=1}^n E_i$$

4. We first show that for any $n \geq 1$,
$$P\left(\bigcup_{i=1}^n A_i\right) \leq \sum_{i=1}^\infty P(A_i).$$
To see this, define
$$B_1 = A_1, \quad B_i = A_i \setminus \left(\bigcup_{j=1}^{i-1} A_j\right), \ i \geq 2.$$
Then
$$P\left(\bigcup_{i=1}^n A_i\right) = P\left(\bigcup_{i=1}^n B_i\right) = \sum_{i=1}^n P(B_i) \leq \sum_{i=1}^n P(A_i) \leq \sum_{i=1}^\infty P(A_i).$$
Now, since this is true for any $n \geq 1$, by the continuity of probability, we have
$$P\left(\bigcup_{i=1}^\infty A_i\right) \leq \sum_{i=1}^\infty P(A_i).$$

5.

$$\begin{aligned}
P\left(\bigcap_{i=1}^{\infty} A_i\right) &= 1 - P\left(\bigcap_{i=1}^{\infty} A_i)^c\right) \\
&= 1 - P\left(\bigcup_{i=1}^{\infty} A_i^c\right) \\
&\geq 1 - \sum_{i=1}^{\infty} P(A_i^c) \\
&= 1 - \sum_{i=1}^{\infty} 0 \\
&= 1
\end{aligned}$$