Assignment-9 of MATH 3270A

November, 2015

P.447

1.

$$\mathbf{x}' = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$$

Solution:
Through all the solutions in this file A denote the constant coefficient matrix of the ODE system and $g(t)$ be the nonhomogeneous term.

$$\begin{aligned} &|\lambda I - A| = \lambda^2 - 1 = 0 \\ &\lambda_{1,2} = \pm 1 \end{aligned}$$

The corresponding eigenvectors are

$$\xi = \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \qquad \eta = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So the fundamental matrix of solution is

$$\mathbf{M}(t) = \begin{pmatrix} -3e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix} \Longrightarrow \mathbf{M}^{-1}(t) = \begin{pmatrix} -\frac{1}{2}e^{-t} & -\frac{1}{2}e^{-t} \\ -\frac{1}{2}e^t & -\frac{3}{2}e^t \end{pmatrix}$$

Use variation of parameter to find one particular solution,

$$\begin{aligned} \mathbf{x}_p(t) =& \mathbf{M}(t) \int \mathbf{M}^{-1}(t) g(t) dt \\ =& \begin{pmatrix} \frac{3}{2}te^t - \frac{1}{4}e^t - 3t \\ -\frac{1}{2}te^t - 1 + 2t + \frac{1}{4}e^t \end{pmatrix} \end{aligned}$$

So the general solution to the system

$$\mathbf{x}(t) = (c_1, c_2)\mathbf{M} + \mathbf{x}_p(t),$$

where c_1 and c_2 are arbitrary constants. □

3.

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ -5 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

Solution:

$$\begin{aligned} &|A - \lambda I| = \lambda^2 + 1 = 0, \\ &\lambda_{1,2} = \pm i. \end{aligned}$$

Corresponding eigenvectors,

$$\xi = \begin{pmatrix} 1 \\ -2 + i \end{pmatrix}, \qquad \eta = \begin{pmatrix} -1 \\ 2 + i \end{pmatrix}.$$

Fundamental matrix of solutions,

$$\mathbf{M(t)} = \begin{pmatrix} \cos t & \sin t \\ -2\cos t - \sin t & \cos t - 2\sin t \end{pmatrix}, \quad \mathbf{M^{-1}(t)} = \begin{pmatrix} \cos t - 2\sin t & -\sin t \\ 2\cos t + \sin t & \cos t \end{pmatrix}.$$

Use variation of parameters to find one particular solution

$$\begin{aligned} \mathbf{x}_p(t) =& \mathbf{M}(t) \int \mathbf{M}^{-1}(t) g(t) dt \\ =& \begin{pmatrix} -t\sin t - t\cos t - \frac{1}{2}\cos t \\ t\cos t + \cos t + 3t\sin t - \frac{1}{2}\sin t \end{pmatrix} \end{aligned}$$

So the general solution is

$$\mathbf{x}(t) = (c_1, c_2)\mathbf{M} + \mathbf{x}_p(t).$$

□

5.

$$\mathbf{x}' = \begin{pmatrix} 4 & 8 \\ -2 & -4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}$$

Solution:

$$\begin{aligned} &|\lambda I - A| = \lambda^2 = 0 \\ &\lambda_{1,2} = 0 \end{aligned}$$

$$\begin{cases} (A - \lambda I)\xi = 0 \\ (A - \lambda I)\eta = \xi \end{cases} \Longrightarrow \xi = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \eta = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}.$$

So the fundamental matrix of solution

$$\mathbf{M} = \begin{pmatrix} 2 & 2t+\frac{1}{2} \\ -1 & -t \end{pmatrix}, \mathbf{M}^{-1} = \begin{pmatrix} -2t & -4t-1 \\ 2 & 4 \end{pmatrix}$$

Use variation of parameter to find one particular solution,

$$\begin{aligned} \mathbf{x}_p(t) =& \mathbf{M}(t) \int \mathbf{M}^{-1} g(t) dt \\ =& \begin{pmatrix} 8 + 8\ln|t| - \frac{1}{2}t^{-2} + 2t^{-1} \\ -4\ln|t| - 4 \end{pmatrix} \end{aligned}$$

So the general solution is

$$\mathbf{x}(t) = (c_1, c_2)\mathbf{M} + \mathbf{x}_p(t).$$

$\square$

7.

$$\mathbf{x}' = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t$$

Solution:

$$\begin{aligned} &|\lambda I - A| = (\lambda - 3)(\lambda + 1) = 0 \\ &\lambda_1 = -1, \lambda_2 = 3. \end{aligned}$$

Corresponding eigenvectors,

$$\xi = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \eta = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

The fundamental matrix and the invertible matrix is

$$\mathbf{M} = \begin{pmatrix} 2e^{-t} & 2e^{3t} \\ -e^{-t} & e^{3t} \end{pmatrix}, \mathbf{M}^{-1} = \begin{pmatrix} \frac{1}{4}e^{t} & -\frac{1}{2}e^{t} \\ \frac{1}{4}e^{-3t} & \frac{1}{2}e^{-3t} \end{pmatrix}$$

Use variation of parameters to find one particular solution,

$$\begin{aligned} \mathbf{x}_p(t) =& \mathbf{M}(t) \int \mathbf{M}^{-1} g(t) dt \\ =& \begin{pmatrix} e^t \\ -\frac{1}{2}e^t \end{pmatrix}. \end{aligned}$$

So the general solution is

$$\mathbf{x}(t) = (c_1, c_2)\mathbf{M} + \mathbf{x}_p(t).$$

$\square$

P.505
(a) Find the eigenvalues and eigenvectors.
(b) Classify the critical point $(0,0)$ as to type, and determine whether it is stable, asymptotically stable, or unstable.
(c) Sketch several trajectories in the phase plane, and also sketch some typical graphs of x_1 versus t.
(d) Use a computer to plot accurately the curves requested in part (c).

1.
$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & 2 \\ -2 & -2 \end{pmatrix} \mathbf{x}$$

Solution:
$\lambda_1 = -1$, $\xi = (1, -2)^T$; $\lambda_2 = 2$, $\eta = (2, -1)^T$; unstable saddle point.
3.
$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \mathbf{x}$$

Solution:
$\lambda_1 = -1$, $\xi = (1, -1)^T$; $\lambda_2 = 1$, $\eta = (3, -1)^T$; unstable saddle point.
7.
$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix} \mathbf{x}$$

Solution:
$\lambda_{1,2} = 1 \pm 2i$, unstable source point.
9.
$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix} \mathbf{x}$$

Solution:
$\lambda_{1,2} = 1$, $\xi = (1, -2)^T$, $\eta = (1, -1)^T$; unstable improper node point.
12.
$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{pmatrix} \mathbf{x}$$

Solution:
$\lambda_{1,2} = \dfrac{1}{2} \pm \dfrac{3}{2}i$, unstable spiral point.