ODE Tut 2

Revision:

Uniqueness and existence thm:

Let $f$ and $\frac{df}{dy}$ be class in some rectangle $\alpha < t < \beta, \gamma < y < \delta$, containing $(t_0, y_0)$, then in some interval $t_0 - h < t < t_0 + h$ in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ satisfying

$$\begin{cases} \frac{dy}{dt} = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

(If the equation is in form of $y' + py = g$, it will have a unique solution if $p$ and $g$ are class).

Def: A equation in form of $y' = f(y)$ is called autonomous,

Def: The constant function satisfies the autonomous ODE is called equilibrium solution and the zeros of $f(y)$ are critical point.
Def: Let \( M(x, y) \, dx + N(x, y) \, dy = 0 \). If \( \exists \, \psi(x, y) \) s.t. \( \frac{\partial \psi}{\partial x} = M \) and \( \frac{\partial \psi}{\partial y} = N \), then \( \psi \) is called an exact ODE.

Thm: If \( M, N, \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x} \) are continuous in the rectangular region \( \alpha < x < \beta, \gamma < y < \delta \), then \( \psi \) is exact if \( My = Nx \).

We can convert a ODE that is not exact into an exact ODE by multiplying a integrating factor.

If \( \mu \) is an integrating factor, then

\[
\frac{\partial \mu}{\partial x} = \left( \frac{My - Nx}{N} \right) \mu \quad \text{or} \quad \frac{\partial \mu}{\partial y} = \left( \frac{Nx - My}{M} \right) \mu.
\]

Try to solve it,

If \( \frac{\partial \mu}{\partial x} = \left( \frac{My - Nx}{N} \right) \mu \), then

\[
\mu M \, dx + \mu N \, dy = 0 \quad \text{is exact.}
\]
\[(l(M)y = l(M)y = N \frac{d^2y}{dx^2} + N_0 \frac{dy}{dx} = (N_0)x\]

Second order ODE

\[P(t)y'' + Q(t)y' + R(t)y = f(t)\]

A 2nd order ODE is homogeneous if \[f(t) = 0\], that is \[P(t)y'' + Q(t)y' + R(t)y = 0\]

If \(P, Q, R\) are constant, say

\[ay'' + by' + cy = 0, \quad a \neq 0\]

**Step 1:** Let the characteristic equation be \[a\lambda^2 + b\lambda + c = 0\], find the roots of \(\lambda\).

**Step 2:** Case 1. \(\lambda = \lambda_1\) or \(\lambda_2\), \(\lambda_1 \neq \lambda_2\), both are real,

the solution is \[y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\], \(c_1, c_2\) are constant.
Case 2 \( \lambda = \lambda_1 \in \mathbb{R} \) double roots,
\[
y = (c_1 + c_2t)e^{\lambda_1 t}
\]
Case 3 \( \lambda_1 = \lambda_2 = \lambda \pm i\alpha \)
\[
y = e^{\lambda t}(C_1 \cos \alpha t + C_2 \sin \alpha t)
\]

Problem

1. Determine if they are exact. If exact, find a solution.

[Q]: \((x^4 + 4y) + (4x - 3y^2)y' = 0\)

\textbf{Ans} : \(\frac{\partial}{\partial y} (x^4 + 4y) = 4\)
\[
\frac{\partial}{\partial x} (4x - 3y^2) = 4, \text{ it is exact,}
\]
\[
\exists \psi \text{ s.t.}, \quad \frac{\partial \psi}{\partial x} = x^4 + 4y
\]
\[
\psi(x,y) = \frac{x^5}{5} + 4xy + h(y)
\]

and \(\frac{\partial \psi}{\partial y} = 4x + h'(y) = 4x - 3y^2\)
\[
h = -\frac{3}{9} y^9 + C
\]
\[
\therefore \psi(x,y) = C \text{ is a solution}
\]
\[
\frac{x^5}{5} + 4xy - \frac{1}{3} y^9 = C
\]
1. \((2xy^2 + 2y) + (2x^2y + 2x)y' = 0\)

**Ans:** \[ \frac{2}{dy} (2xy^2 + 2y) = 4xy + 2 \]
\[ \frac{2}{dx} (2x^2y + 2x) = 4xy + 2 \]

It is exact

So \[ \frac{2y}{dx} = 2xy^2 + 2y \]

\[
\psi = x^2y^2 + 2xy + h(y)
\]

\[ \frac{2y}{dy} = 2x^2y + 2x + h'(y) = 2x^2y + 2x \]

\[ h = C \]

\[ x^2y^2 + 2xy = C. \]

2. Use integrating factor to solve,

1a \((3x^2y + 2xy + y^3) \ dx + (x^2 + y^2) \ dy = 0\)

**Ans:** \[ \frac{2}{dy} (3x^2y + 2xy + y^3) = 3x^2 + 2x + 3y^2 \]

\[ \frac{2}{dx} (x^2 + y^2) = 2x \]

It is not exact,

\[ \frac{au}{dx} = \frac{3x^2 + 2x + 3y^2 - 2x}{x^2 + y^2} \]

\[ u = 3 \]

\[ \therefore u = e^{3x} \] (Constant is not needed).
Multiple $e^{3x}$,

$$e^{3x} (3x^2 y + 2xy + y^3) \, dx + e^{3x} (x^2 + y^2) \, dy = 0$$

$$\exists \psi, \text{ s.t. } \frac{\partial \psi}{\partial x} = e^{3x} (3x^2 y + 2xy + y^3)$$

$$\psi = 3 y \int e^{3x} x^2 \, dx + 2 y \int e^{3x} x \, dx + y^3 \int e^{3x} \, dx + h(y)$$

$$= y \int x^2 \, dx e^{3x} + 2 y \int e^{3x} x \, dx + y^3 \frac{e^{3x}}{3} + h(y)$$

$$= y (x^2 e^{3x} - 2 \int e^{3x} x \, dx) + 2 y \int e^{3x} x \, dx + y^3 \frac{e^{3x}}{3} + h(y)$$

$$= y x^2 e^{3x} + y^3 \frac{e^{3x}}{3} + h(y).$$

and

$$\frac{\partial \psi}{\partial y} = x^2 e^{3x} + y^2 e^{3x} + h'(y) = e^{3x} (x^2 + y^2)$$

$$n = C$$

$$\therefore \psi = C$$

$$y x^2 e^{3x} + y^3 \frac{e^{3x}}{3} = C$$
\[ \frac{dy}{dx} + \left( \frac{x}{y} - \sin y \right) dy = 0, \]

**Ans:** \[ \frac{\partial u}{\partial y} = \frac{1}{y} \quad u = \frac{1}{y} \]

\[ \therefore \quad u = y \]

\[ y dx + (x - y \sin y) dy = 0. \]

\[ \psi \leq 2 \quad \frac{\partial \psi}{\partial x} = y \]

\[ \psi = xy + h(y) \]

\[ \frac{\partial \psi}{\partial y} = x + h'(y) = x - y \sin y \]

\[ h'(y) = -y \sin y \]

\[ h = \int y \cos y dy = y \cos y - \sin y + C. \]

\[ \psi = C \]

\[ xy + y \cos y - \sin y = C \]
3 Solve
\[ \frac{a y}{a t} = \frac{y^{1993} \cos e^{(t^{10} + y^{20})^7}}{1 + t^4 + y^8} \]
\[ y(0) = 0. \]
Ans: since \[ y^{1993} \cos e^{(t^{10} + y^{20})^7} \]
\[ 1 + t^4 + y^8 \]
smooth, its derivative must be cts.
by uniqueness and existence thm,
\[ y = 0. \]

4 Solve \[ y'' + 5y' + 6y = 0 \]
Ans:
\[ \lambda^2 + 5\lambda + 6 = 0 \]
\[ \lambda = -2 \text{ or } -3 \]
\[ y = c_1 e^{-2t} + c_2 e^{-3t} \]

5 Solve
\[ \begin{cases} 
4y'' - 4y' + y = 0 \\
y(0) = 2 \\
y'(0) = \frac{1}{3}
\end{cases} \]
Ans:
\[ 4\lambda^2 - 4\lambda + 1 = 0 \]
\[ \lambda = \frac{1}{2} \]
\[ y = (c_1 t + c_2) e^{\frac{t}{2}} \]
\[ y(0) = c_2 = 2 \]
\[ y'(t) = \frac{1}{2} e^{\frac{t}{2}} (c_1 t + c_2) + e^{\frac{t}{2}} c_1 \]
\[ y'(0) = 1 + c_1 = \frac{1}{3} \]
\[ c_1 = -\frac{2}{3} \]
\[ \therefore y = 2 e^{\frac{t}{2}} - \frac{2}{3} t e^{\frac{t}{2}} \]

6. Solve \( y'' + y' + y = 0 \)

\textbf{Ans:} \[ \lambda^2 + \lambda + 1 = 0 \]
\[ \lambda = \frac{-1 \pm \sqrt{3} i}{2} \]
\[ \therefore y = e^{\frac{t}{2}} \left( c_1 \sin \left( \frac{\sqrt{3}}{2} t \right) + c_2 \cos \left( \frac{\sqrt{3}}{2} t \right) \right) \]