ODE TUT 1

① $\frac{dy}{dt} + \frac{1}{4}y = 3 + 2\cos t$ , $y(0) = 0$

A: Integrating factor $= \mu = e^{\int \frac{1}{4}} = e^{\frac{t}{4}}$

$$\frac{dy}{dt}e^{\frac{t}{4}} + \frac{ye^{\frac{t}{4}}}{4} = (3 + 2\cos t)e^{\frac{t}{4}}$$

$$\frac{d(y \cdot e^{\frac{t}{4}})}{dt} = (3 + 2\cos t)e^{\frac{t}{4}}$$

$$ye^{\frac{s}{4}} = 12e^{\frac{s}{4}} + 2\int_0^s \cos t e^{\frac{t}{4}} dt$$

And $\int_0^s \cos t e^{\frac{t}{4}} dt = 4\int_0^s \cos t \, de^{\frac{t}{4}}$

$$= 4(\cos s e^{\frac{s}{4}} - 1) + 4\int_0^s e^{\frac{t}{4}} \sin t dt$$

$$= 4(\cos s e^{\frac{s}{4}} - 1) + 16\int_0^s \sin t \, de^{\frac{t}{4}}$$

$$= 4(\cos s e^{\frac{s}{4}} - 1) + 16(\sin s e^{\frac{s}{4}} - \int_0^s e^{\frac{t}{4}} \cos t dt)$$

$$\therefore \int_0^s \cos t e^{\frac{t}{4}} dt = \frac{4(\cos s e^{\frac{s}{4}} - 1)}{17} + \frac{16 \sin s e^{\frac{s}{4}}}{17}$$

$$y = 12 + \frac{8}{17}\cos t + \frac{32}{17}\sin t - \frac{8}{17}e^{-\frac{1}{4}t}$$

② $\frac{dy}{dx} + \frac{y}{x} = y^2$

A: Sub $y = \frac{1}{z}$,

If $x > 0$, $\frac{dy}{dx} = -\frac{dz}{dx}\left(\frac{1}{z^2}\right)$

$$-\frac{dz}{dx}\left(\frac{1}{z^2}\right) + \frac{1}{zx} = \frac{1}{z^2}$$

$$\frac{dz}{dx} - \frac{z}{x} = -1$$

$$\mu = e^{\int -\frac{1}{x}} = e^{-\ln x} = \frac{1}{x}$$

$$\frac{dz}{dx}\left(\frac{1}{x}\right) - \frac{z}{x^2} = \frac{-1}{x}$$

$$\frac{dz}{dx}\left(\frac{1}{x}\right) - \frac{z}{x^2} = -\frac{1}{x}$$

$$\frac{d}{dx}\left(\frac{z}{x}\right) = \frac{-1}{x}$$

$$\frac{z}{x} = -\ln x + C$$

$$z = -x\ln x + Cx$$

$$y = \frac{1}{x(C - x\ln x)}.$$

there is no solution if $x = 0$,

But there is solution if $x < 0$.

③ $\frac{dy}{dx} = \frac{ax+by}{cx+dy}$, $a, b, c, d \in \mathbb{R} \setminus \{0\}$.

A: sub $w = \frac{y}{x}$, $y = xw$,

$$\frac{dy}{dx} = \frac{dw}{dx}x + w = \frac{ax+by}{cx+dy}$$

$$= \frac{a+bw}{c+dw}$$

$$\frac{c+dw}{a+(b-c)w-dw^2}\frac{dw}{dx} = \frac{1}{x}$$

using completing the square, you will get the answer.

$$\frac{(b+c)}{(-b^2+2bc-c^2-4ad)^{\frac{1}{2}}}\tan^{-1}\left(\frac{-b+c+\frac{2dy}{x}}{(-b^2+2bc-c^2-4ad)^{\frac{1}{2}}}\right)+$$

$$\frac{1}{2}\ln\left(-ax^2+(-b+c)xy+dy^2\right)=C \quad , C\in\mathbb{R}$$

④ $\frac{dy}{dx}=\frac{x^2-3y^2}{2xy}$

If $x>0$, let $y=xw$,

$$xw'+w=\frac{1}{2w}-\frac{3w}{2}$$

$$xw'=\frac{1}{2w}-\frac{5w}{2}$$

$$\int\frac{2w\,dw}{1-5w^2}=\int\frac{dx}{x}$$

$$\int\frac{d(\ln|1-5w^2|)}{-5}=\ln x+C$$

$$\ln|1-5w^2|=-5\ln x+C$$

$$w=\pm\sqrt{\frac{1+Cx^{-5}}{5}}$$

$$y=\pm x\sqrt{\frac{1+Cx^{-5}}{5}}\,.$$

Simillar with question ②

⑤ A tank contains 100 gallons of water and 50 oz of salt. Water containing salt of concentration of $\frac{1}{4}(1+\frac{1}{2}\sin t)$ oz/gal flow into the tank at rate of 2 gal/min and the mixture flow out at same rate

5a find amount of salt in tank at time t,

5b what is the long time behavior?

5a: $\frac{dx}{dt} = in - out$

$$= 2\left(\frac{1}{4}\right)\left(1+\frac{1}{2}\sin t\right) - 2\left(\frac{x}{100}\right) \quad \left(\begin{array}{c}\text{speed} \times \\ \text{concentration}\end{array}\right)$$

$$\frac{dx}{dt} + \frac{x}{50} = \frac{1}{2}\left(1+\frac{1}{2}\sin t\right)$$

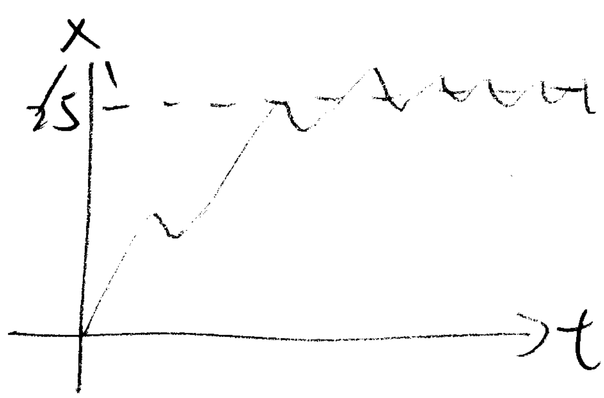
$$\mu = e^{\int \frac{1}{50}} = e^{\frac{t}{50}},$$

$$\frac{dx}{dt}e^{\frac{t}{50}} + \frac{x}{50}e^{\frac{t}{50}} = \frac{1}{2}\left(1+\frac{1}{2}\sin t\, e^{\frac{t}{50}}\right)$$

$$\frac{d\left(xe^{\frac{t}{50}}\right)}{dt} = \frac{1}{2}\left(1+\frac{1}{2}\sin e^{\frac{t}{50}}\right)$$

$$x = \frac{63150}{2501}e^{-\frac{t}{50}} + 25 - \frac{625}{2501}\cos t + \frac{25}{2501}\sin t$$

5b It will oscillate at $x=25$ with amplitude about $\frac{625}{2501}$



Q $x^2y'' - xy' + y = 0$ (Hint: reduce to 1st order)

A: sub $y = xv$

$$y' = xv' + v$$
$$y'' = xv'' + 2v'$$
$$x^2(xv'' + 2v') - x(xv' + v) + xv = 0$$
$$x^3 v'' + x^2 v' = 0$$

let $v' = u$

$$u' + \frac{1}{x}u = 0$$
$$\mu = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$
$$xu' + u = 0$$
$$\frac{d}{dx}(xu) = 0$$
$$xu = C_1$$
$$u = \frac{C_1}{x}$$
$$v = C_1 \ln|x| + C_2$$
$$y = x(C_1 \ln|x| + C_2)$$