11.1 Zero Integral on Closed Contour

We continue to discuss the situations where $\int_{\Gamma} f = 0$ for a closed (usually simple) contour. Previously, we have two conditions; each one is sufficient to guarantee a zero integral.

1. $F'(z) = f(z)$ for $z \in \Omega$ and $\Omega \supset \Gamma$. (**)
2. $f$ is of $C^1$ and satisfies Cauchy-Riemann on $\Omega$; also $\Gamma \cup S_b \subset \Omega$. (††)

Let us focus on the second one (††). At the beginning, we know that there is a close relation between complex differentiability and Cauchy-Riemann Equations.

The interesting and surprising part about complex functions is that the above diagram will be changed when it is true at every point $z_0 \in \Omega$. And the proof indeed goes through contour integration. More precisely, the dotted implications in the diagram below can be proved.

In this lesson, we will discuss the proof of Goursat. It involves very detail estimate and $\varepsilon$-$\delta$ argument. One is encouraged to understand the overall logic flow before studying the analysis.
11.1.1 Integrating analytic functions

Let us first understand the situation when the integrand \( f \) is analytic. Near a point \( z_0 \), we have

\[
    f(z) = f(z_0) + f'(z_0)(z - z_0) + \text{(small error)}, \quad \lim_{z\to z_0} \frac{\text{small error}}{z - z_0} = 0.
\]

Its integral along a small contour \( \gamma \) around \( z_0 \) is given by

\[
    \int_\gamma f = \int_\gamma f(z_0) + \int_\gamma f'(z_0)(z - z_0) + \int_\gamma \text{(small error)} dz = \int_\gamma \text{(small error)} dz,
\]

because of existence of antiderivatives for the first two integrals. Thus, \( \text{(small error)} \) is the crucial content and we also expect that the third integral is small. In the example below, we will investigate the situation of integrating a \( \text{(small error)} \) along a contour in a small region. For this, we will use the notation

\[
    \text{(small error)} = \eta_0(z)(z - z_0), \quad \text{where} \quad \lim_{z\to z_0} \eta_0(z) = 0.
\]

**Example 11.1.** In this example, we deal with two cases of contours that occur over a small region. The first case is the contour \( \partial \Box = (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \) formed by the four sides of a square \( \Box \) with center \( z_0 \) and side length \( 2\delta \) (left hand picture). The second case is a contour \( \Gamma_0 \) with an arc \( \gamma \) inside \( \Box \) and some parts \( \sigma_1, \ldots, \sigma_4 \) along the sides of \( \Box \) (right hand picture).

Let \( \eta \) be a function as above, i.e., defined on an open set containing \( \Box \cup \partial \Box \) with \( |\eta(z)| \leq \varepsilon \) for all \( z \in \Box \cup \partial \Box \). We are going to give an upper bound for the integrals of the above two cases, namely, \( \int_{\partial \Box} \eta(z)(z - z_0) dz \) and \( \int_{\Gamma_0} \eta(z)(z - z_0) dz \).

Since \( \sigma_1 \) is from \( z_0 + \delta(1 - i) \) to \( z_0 + \delta(1 + i) \), it can be parametrized by

\[
    z(t) = (1 - t)[z_0 + \delta(1 - i)] + t[z_0 + \delta(1 + i)], \quad t \in [0, 1].
\]

We have \( z'(t) = \delta(1 + i) - \delta(1 - i) = 2\delta i \). Together with \( |\eta(z(t))| \leq \varepsilon \), we have

\[
    \left| \int_{\sigma_1} \eta(z)(z - z_0) dz \right| \leq \int_0^1 |\eta(z(t))| \cdot |z(1) - z(0)| \cdot |2\delta i| \, dt \\
    \leq \int_0^1 \varepsilon \cdot (\sqrt{2}\delta) \cdot (2\delta) \, dt = 2\sqrt{2}\delta^2 \varepsilon.
\]

Thus

\[
    \left| \int_{\partial \Box} \eta(z)(z - z_0) dz \right| \leq 8\sqrt{2}\delta^2 \varepsilon = 2\sqrt{2} \text{Area}(\Box)\varepsilon.
\]
The contour $\Gamma_0$ may almost include all four sides $\partial \square$ of the square and a curve $\gamma$ inside the square $\square$. Then,

$$\left| \int_{\Gamma_0} \eta(z)(z - z_0)\,dz \right| \leq \sum_{k=1}^{4} \left| \int_{\sigma_k} \eta(z)(z - z_0)\,dz \right| + \left| \int_{\gamma} \eta(z)(z - z_0)\,dz \right|$$

$$\leq 2\sqrt{2} \text{Area}(\square)\varepsilon + \varepsilon \int_{\gamma} |(z - z_0)\,dz|$$

$$\leq 2\sqrt{2} \text{Area}(\square)\varepsilon + \sqrt{2}\delta \text{Length} (\gamma) \varepsilon .$$

### 11.1.2 Cauchy-Goursat

**Theorem 11.2.** Let $\Gamma$ be a simple closed contour with bounded complement component $S_b$ such that $\Gamma \cup S_b \subset \Omega$ and $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is analytic. Then $\int_{\Gamma} f = 0$.

The result of Example 11.1 is very useful in understanding the proof of the theorem. We are going to divide $\Gamma \cup S_b$ into small squares $\square_k$ of side length $2\delta$ as shown in the figure below.

Given any $\varepsilon > 0$, by the analyticity of $f$ on $\Omega$, there exists a suitable $\delta > 0$ (the compactness of $\Gamma \cup S_b$ is needed) such that if $z_0 \in \Gamma \cup S_b$ and $z \in \Omega$ with $|z - z_0| < \sqrt{2}\delta$, then

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta_0(z)(z - z_0) \quad \text{where} \quad |\eta_0(z)| \leq \varepsilon .$$

Further reduce $\delta$ to be small enough such that there is a subset $K$ such that $\Gamma \cup S_b \subset K \subset \Omega$. The set $K$ is a union of squares $\square_k$, $k = 1, \ldots, N$, such that each $\square_k$ either lies in $S_b$ completely (case 1 in Example 11.1) or intersects $\Gamma$ (case 2 above). Denote $\gamma_k = \square_k \cap \Gamma$, which plays the role of $\gamma$ in Example 11.1.

It can be easily observed that

$$\sum_{k=1}^{N} \int_{\partial \square_k} f(z)\,dz = \int_{\partial K} f(z)\,dz \neq \int_{\Gamma} f(z)\,dz .$$
By the argument of the Example 11.1 (note that $\gamma_k = \emptyset$ if $k \subset S_b$),

$$\left| \int_{\Gamma} f(z) \, dz \right| \leq \sum_{k=1}^{N} \left| \int_{(\partial \square_k \cap \gamma_k)} f(z) \, dz \right| = \sum_{k=1}^{N} \left| \int_{(\partial \square_k \cap \gamma_k)} \eta_k(z) (z - z_k) \, dz \right|
$$

$$\leq \sum_{k=1}^{N} \left( \int_{\partial \square_k} \eta_k(z) (z - z_k) \, dz \right) + \left| \int_{\gamma_k} \eta_k(z) (z - z_k) \, dz \right|
$$

$$\leq \sum_{k=1}^{N} \left[ 2\sqrt{2} \text{Area}(\square_k) \varepsilon + \sqrt{2} \delta \text{Length}(\gamma_k) \varepsilon \right]
$$

$$\leq \sqrt{2} \left[ 2 \text{Area}(K) + \text{Length}(\partial K) \text{Length}(\Gamma) \right] \varepsilon
$$

$$\leq \sqrt{2} \left[ 4 \text{Area}(S_b) + 2 \text{Length}(\Gamma)^2 \right] \varepsilon.
$$

Since $\varepsilon > 0$ is arbitrary, we have $\left| \int_{\Gamma} f(z) \, dz \right| = 0$.

### 11.2 Further Results

In the statement of Cauchy-Goursat Theorem, the contour is simple closed. However, the result is useful in many other situations. Furthermore, the statement apparently is about analytic functions, it actually is often used on integrand that are not analytic.

**Example 11.3.** Let $\Gamma_1$ be a circle of center 0 and radius $R$; $\Gamma_2$ be the contour as shown.

![Diagram](image)

What are $\int_{\Gamma_1} g$ and $\int_{\Gamma_2} g$ in the cases that $g(z) = 1/z^3$ or $g(z) = 1/z$?

Since $\Gamma_1$ can be explicitly parametrized by $Re^{it}$ for $t \in [0, 2\pi]$, we may directly calculate that

$$\int_{\Gamma_1} \frac{1}{z^3} \, dz = 0 \quad \text{and} \quad \int_{\Gamma_1} \frac{1}{z} \, dz = 2\pi i.
$$

Without explicit parametrization for $\Gamma_2$, can we use Cauchy-Goursat Theorem to get the result? The answer is no; because for both $\Gamma_1$ and $\Gamma_2$, the bounded complement component $S_b$ contains 0 and thus $g$ is not analytic on $\Gamma \cup S_b$. Still, we have

$$\int_{\Gamma_2} \frac{1}{z^3} \, dz = 0 \quad \text{because } g(z) = 1/z^3 \text{ has an antiderivative on } \mathbb{C} \setminus \{0\} \supset \Gamma_2.
$$

Obviously, for $g(z) = 1/z$, it does not have an antiderivative on $\mathbb{C} \setminus \{0\}$, one must look for another method. Let $\gamma$ be a curve (not necessarily straight) joining $\Gamma_1$ and $\Gamma_2$.

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Then the contour \( \Gamma = (-\Gamma_1, \gamma, \Gamma_2, -\gamma) \) actually satisfies \( \partial S_b = \Gamma \). Moreover, the proof of Cauchy-Goursat Theorem is still valid in such a situation. Therefore, we have

\[
0 = \int_{\Gamma} f(z) \, dz = \int_{\Gamma_2} f + \int_{\gamma} f - \int_{\gamma} f - \int_{\Gamma_1} f,
\]

and thus
\[
\int_{\Gamma_2} \frac{1}{z} \, dz = \int_{\Gamma_1} \frac{1}{z} \, dz = 2\pi i.
\]

### 11.2.1 Regions with Holes

As in Example 11.3 above, we always need to deal with functions that are not completely analytic. Thus, it is convenient to have a more general version of Cauchy-Goursat Theorem.

Let \( \Gamma_0 \) be a simple closed contour and \( \Gamma_1, \ldots, \Gamma_p \) be simple closed contours contained in the bounded complement component of \( \Gamma_0 \) (see an illustration below).

Let \( S_{b,0}, S_{b,1}, \ldots, S_{b,p} \) be the bounded complement components of \( \Gamma_0, \Gamma_1, \ldots, \Gamma_p \) respectively. Then \( B = S_{b,0} \cap \bigcap_{k=1}^{p} (\mathbb{C} \setminus S_{b,k}) \) is the region between the contours.

**Theorem 11.4.** In the setting above, let \( \Gamma_0 \) and all \( \Gamma_k, \ k = 1, \ldots, p \) be positively oriented. If \( f : \Omega \subset \mathbb{C} \to \mathbb{C} \) is analytic on a domain \( \Omega \supset B \), then

\[
\int_{\Gamma_0} f = \sum_{k=1}^{p} \int_{\Gamma_k} f.
\]

**Idea of Proof.** Add short arcs \( \gamma_1, \ldots, \gamma_p \) to connect each contour \( \Gamma_k \) from \( \Gamma_0 \) and then use the argument of Example 11.3.

**Remark.** Note that if all the \( \Gamma_1, \ldots, \Gamma_p \) take negative orientation, then their normals will behave the same as the normal of \( \Gamma_0 \) to point towards \( B \). Corresponding to the above picture, this is written as

\[
\partial B = \Gamma_0 \cup (-\Gamma_1) \cup \cdots \cup (-\Gamma_p).
\]

The situation of Example 11.3 is often expressed in the following form.
THEOREM 11.5 (Invariance of Deformation). If \( \Gamma_1 \) and \( \Gamma_2 \) can be deformed smoothly to each other through a region \( B \) where \( f \) is analytic, then

\[
\int_{\Gamma_1} f = \int_{\Gamma_2} f.
\]

The function \( f \) may not be analytic at many places. However, as long as it is analytic on the yellow region between the two contours, its integrals along the contours are the same.

EXAMPLE 11.6. Let us consider the rational function \( g(z) = \frac{z^3 + z^2 + z - 2}{z^2 - z} \). It is clearly analytic on \( \mathbb{C} \setminus \{0,1\} \). The first observation is that the numerator is of degree 3 while the denominator of degree 2. We may use long division to get

\[
g(z) = z + 2 + \frac{3z - 2}{z^2 - z}.
\]

Thus, for any closed contour \( \Gamma \), we can use Cauchy-Goursat Theorem to have

\[
\int_{\Gamma} g(z) \, dz = \int_{\Gamma} (z + 2) \, dz + \int_{\Gamma} \frac{3z - 2}{z^2 - z} \, dz = 0 + \int_{\Gamma} \frac{3z - 2}{z^2 - z} \, dz.
\]

This illustrates that we can always throw away the analytic part of a function when doing integral along a closed contour. Next, we will use partial fraction to have

\[
\int_{\Gamma} g(z) \, dz = \int_{\Gamma} \frac{3z - 2}{z^2 - z} \, dz = \int_{\Gamma} \left( \frac{2}{z} + \frac{1}{z - 1} \right) \, dz.
\]

For the following contours, we are able to reduce the calculation to the special contours \( C_0 \) and \( C_1 \), which are circles at centers 0 and 1 respectively. The radii of the circles are not important. We assume all contours are positively oriented.

By Invariance of Deformation, \( g \) is analytic on the region between \( \Gamma_1 \) and \( C_1 \), so

\[
\int_{\Gamma_1} g(z) \, dz = \int_{C_1} g(z) \, dz = \int_{C_1} \frac{2}{z} \, dz + \int_{C_1} \frac{1}{z - 1} \, dz.
\]
Note that the integrand $2/z$ is analytic on and inside $C_1$, therefore $\int_{C_1} \frac{2}{z} \, dz = 0$. Moreover, by the parametrization $C_1$ by $z(t) = 1 + Re^{it}$, $t \in [0, 2\pi]$, we may calculate that $\int_{C_1} \frac{1}{z - 1} \, dz = 2\pi i$. Similarly, we have 

$$\int_{\Gamma_2} g(z) \, dz = 4\pi i + 2\pi i = 6\pi i.$$ 

For the contour $\Gamma_3$, we have to break it into simple closed contours and obtain the answer $2\pi i$. This example is easy because it is a fraction of two polynomial. What if $g(z) = \frac{\sin z}{z^2 - z}$? We can certainly use partial fraction to have $g(z) = \left(\frac{-1}{z} + \frac{1}{z - 1}\right)\sin z$. From this, we see a major concern in finding contour integral, namely, the integrand is of the form 

$$g(z) = \frac{f(z)}{z - z_0}$$ 

where $f$ is an analytic function.

This will be our next topic.