THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2230A (First term, 2015–2016)
Complex Variables and Applications
Notes 10 Zero Contour Integrals

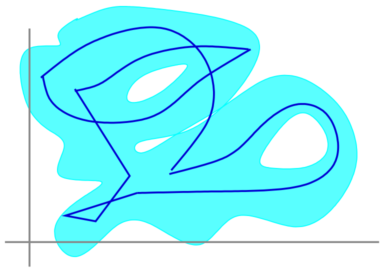
10.1 On Closed Contours

A major question in doing complex contour integration is this:

If the contour Γ is closed, when will $\displaystyle\int_\Gamma f = 0$.

This is important because it is related to that the integral is independent of the contour (but only the end points).

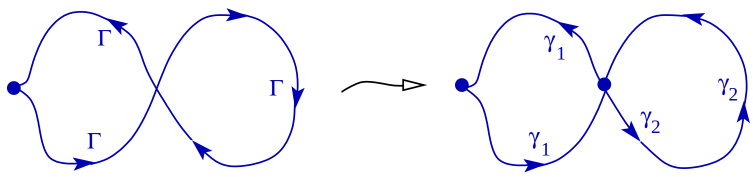
It is known previously that if there exists an antiderivatives F on some domain $\Omega \supset \Gamma$ for f, i.e., $F'(z) = f(z)$ for each $z \in \Omega$, then we have a zero contour integral. ***Note*** that in this situation, both Γ and Ω can have "bad shape", i.e. self-intersections and holes.



On the other hand, the requirement on f that an antiderivative exists is quite *demanding*. It is not easy to check this condition of f unless one is able to find the antiderivative.

In this notes, we will provide a different situation that the contour integral on a closed contour is zero. The condition on the integrand f will be weaker; but in place of this, we need a stronger condition on the shapes of the contour Γ and the domain Ω.

First, to simplify the discussion, we may break contours with self-intersection into simple ones. For example, we may write $\Gamma = (\gamma_1, -\gamma_2)$ for the picture below.



In this case, both γ_1 and γ_2 are *simple closed* contours and

$$\int_\Gamma f = \int_{\gamma_1} f - \int_{\gamma_2} f\,.$$

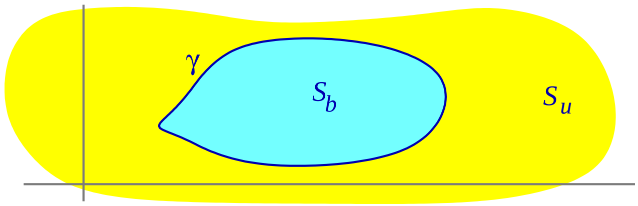
10.1.1 About Simple Contours

As mentioned above, we need a stronger condition involving the contour and the domain. This condition is easier to express if the contour is simple. Later, even when the contour is not simple, the result can be suitably used.

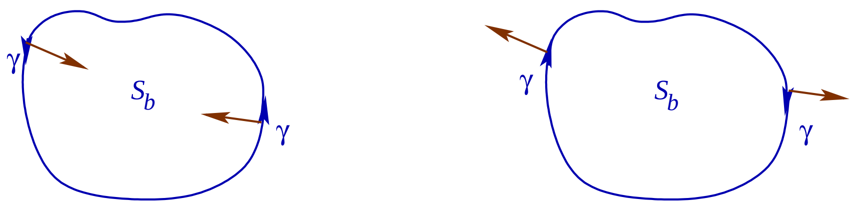
According to the Jordan Curve Theorem, a simple closed contour separates the plane into two pieces, one is bounded and the other is unbounded. Mathematically, if γ is a simple closed contour, then

$$\mathbb{C} \setminus \gamma = S_b \cup S_u \,,$$

where both S_b and S_u are open connected sets such that S_b is bounded and S_u is unbounded.



There is a sign convention in relation to the integral. That is indeed the compatibility of the orientation of the contour and the bounded domain. Let γ be a simple closed contour, its orientation (direction) is given by the parametrization. Its normal is on the left-hand side of the tangent, i.e., the tangent and normal form a positive basis for the plane. The bounded component S_b is either on the normal side or the opposite side of γ as in the figure.



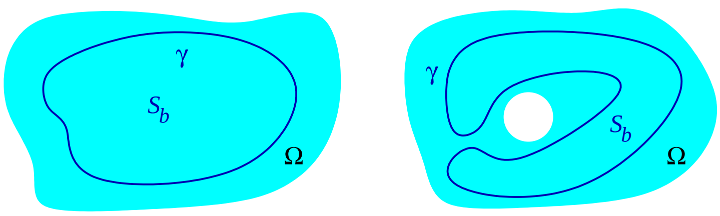
$\partial S_b = \gamma$ $\qquad\qquad$ $\partial S_b = -\gamma$

If the bounded component S_b is on the side of the normal, then γ is positively oriented and denote $\partial S_b = \gamma$; otherwise $\partial S_b = -\gamma$.

A good condition about a simple closed contour γ with bounded S_b and a domain Ω is

$$\boxed{\Omega \supset \gamma \cup S_b} \tag{$**$}$$

This condition is illustrated in the figure below. Note that Ω itself may have holes, but not S_b.



In the left-hand picture, Ω is simply connected. The rigorous definition requires algebraic topology. Intuitively, it means that there is no hole nor puncture. If Ω is simply connected, then for each $\gamma \subset \Omega$, the condition $(**)$ is automatically satisfied. Therefore, some books assume that Ω is simply connected.

10.1.2 Another Useful Result

Now, we are ready to discuss the result that we are looking for. Recall that we want to relax the condition on the integrand f and perhaps strengthen the condition on the domain and contour.

THEOREM 10.1. *Let $f: \Omega \subset \mathbb{C} \to \mathbb{C}$ be of C^1 and satisfy the Cauchy-Riemann Equations on Ω. Then for each simple closed contour γ with bounded complement domain S_b, i.e., $\partial S_b = \pm\gamma$ such that $\gamma \cup S_b \subset \Omega$,*

$$\int_\gamma f = 0 \, .$$

Note that this theorem guarantees the zero answer for a special type of closed contour. The requirement on the function f becomes C^1 and the Cauchy-Riemann Equations. Such requirement implies that f is analytic on Ω. At this point, we do not know whether analyticity is enough for zero integral. Below, if we put a condition on the domain Ω, then it works for any closed contour.

COROLLARY 10.2. *Let Ω be a simply connected domain such that $f: \Omega \subset \mathbb{C} \to \mathbb{C}$ is of C^1 and satisfies the Cauchy-Riemann Equation. Then, for every closed contour $\Gamma \subset \Omega$, we have*

$$\int_\Gamma f = 0 \, .$$

Note that Γ needs not be simple in the corollary because it can be broken into simple pieces.

Main Idea of proof. Write $f(x + \mathbf{i}y) = u(x, y) + \mathbf{i}v(x, y)$ and let $z(t) = x(t) + \mathbf{i}y(t)$, $t \in [a, b]$ be a parametrization of Γ. Without loss of generality, we may assume that $\partial S_b = \Gamma$. Then

$$\begin{aligned}
\int_\Gamma f &= \int_a^b \left[\, u(x(t), y(t)) + \mathbf{i}v(x(t), y(t)) \,\right] \cdot \left[x'(t) + \mathbf{i}y'(t)\right] \, dt \\
&= \int_a^b \left[\, ux' - vy' \,\right] \, dt + \mathbf{i} \int_a^b \left[\, uy' + vx' \,\right] \, dt \\
&= \int_\Gamma (u \, dx - v \, dy) + \mathbf{i} \int_\Gamma (v \, dx + u \, dy) \\
&= \iint_{S_b} (u_x - v_y) \; dxdy + \mathbf{i} \iint_{S_b} (v_x + u_y) \; dxdy \qquad \text{by Green's Theorem.}
\end{aligned}$$

The two double integrals in the last line are zero because of the Cauchy-Riemann Equations. ☐

Remark. In the above proof, in order to use the Green's Theorem, we need the continuity of the functions u_x, u_y, v_x, v_y and so f has to be of C^1. Moreover, we need the contour Γ to be piecewise C^1, which we always assume the contour to be. This version of the theorem is proved by Cauchy. Later, Goursat extend it to a more powerful theorem.

Finally, we now have two conditions that will lead to a zero contour integral on closed contours. One requires the integrand f to have an antiderivative, which is hard to verify. The other requires that f to be C^1 and satisfy the Cauchy-Riemann equations. This is obviously easier to follow. To gain this benefit, we need to put a stronger requirement on the situation of the domain or on the contour.