# Math 2040A Tutorial Week 8

## §6.1 Inner products

2. $x = (2, 1+i, i)$ $y = (2-i, 2, 1+2i)$ are vectors in $\mathbb{C}^3$.
Compute $\langle x, y\rangle$, $\|x\|$, $\|y\|$, $\|x+y\|$

Solution: $\langle x, y\rangle = 2\cdot(2+i) + (1+i)\cdot 2 + i(1-2i) = 8+5i$

$\|x\| = \langle x, x\rangle^{\frac{1}{2}} = (4+2+1)^{\frac{1}{2}} = \sqrt{7}$

$\|y\| = \langle y, y\rangle^{\frac{1}{2}} = (5+4+5)^{\frac{1}{2}} = \sqrt{14}$.

$\|x+y\| = \|(4-i, 3+i, 1+3i)\| = (17+10+10)^{\frac{1}{2}} = \sqrt{37}$.

Cauchy-Schwartz: $|8+5i| = \sqrt{89} \le \sqrt{7}\cdot\sqrt{14}$

Triangle inequality: $\sqrt{7}+\sqrt{14} \ge \sqrt{37}$

5. In $\mathbb{C}^2$, show that $\langle x, y\rangle = xAy^*$ is an inner product, where $A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$ Compute $\langle x, y\rangle$ for $x = (1-i, 2+3i)$ and $y = (2+i, 3-2i)$

Solution: $\langle\cdot,\cdot\rangle$ is an inner product:

- $\langle x+z, y\rangle = (x+z)Ay^* = xAy^* + zAy^* = \langle x, y\rangle + \langle z, y\rangle$
- $\langle cx, y\rangle = cxAy^* = c(xAy^*) = c\langle x, y\rangle$.
- $\overline{\langle x, y\rangle} = (xAy^*)^* = yA^*x^* = yAx^* = \langle y, x\rangle$ $(A = A^*)$
- $\langle x, x\rangle = (x_1, x_2)A(x_1, x_2)^* = \|x_1\|^2 + 2\,\mathrm{Re}(ix_1\bar{x}_2) + 2\|x_2\|^2 > 0$ (if $x_1$ or $x_2$ is not 0)

$\langle x, y\rangle = 6+12i$

8. Why the following are not an inner product?

a) $\langle (a,b), (c,d)\rangle = ac - bd$ on $\mathbb{R}^2$

b) $\langle A, B\rangle = tr(A+B)$ on $M_{2\times 2}(\mathbb{R})$

c) $\langle f(x), g(x)\rangle = \int_0^1 f'(t)g(t)$ on $P(\mathbb{R})$.

Solution: a) $\langle (1,1), (1,1)\rangle = 0$

b) $A = B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\langle 2A, B\rangle = 3$ but $2\langle A, B\rangle = 4$

c) $f(x) = 1$, $g(x) = f(x) = 1$, $\langle f(x), g(x)\rangle = 0$

10. V – an inner product space, and suppose that $x$ and $y$ are orthogonal vectors in V. Prove that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. Deduce the Pythagorean theorem in $\mathbb{R}^2$.

Solution: $\|x+y\|^2 = \langle x+y, x+y \rangle = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$

$= \langle x, x \rangle + \langle y, y \rangle = \|x\|^2 + \|y\|^2$

(Note that $x$ and $y$ are orthogonal $\Leftrightarrow$ $\langle x, y \rangle = 0$)

To deduce Pythagorean theorem in $\mathbb{R}^2$. note we have the following graph:


$AC^2 = \|x\|^2$ $AB = \|y\|$ $BC = \|x+y\|$

Thus $AC^2 + AB^2 = BC^2$

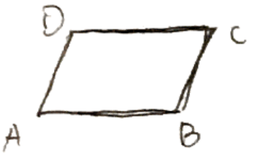
11. Prove the parallelogram law on the inner product space V:

$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in V$.

Solution: $\|x+y\|^2 + \|x-y\|^2 = (\|x\|^2 + \langle x, y \rangle + \langle y, x \rangle + \|y\|^2) +$

$(\|x\|^2 - \langle x, y \rangle - \langle y, x \rangle + \|y\|^2)$

$= 2\|x\|^2 + 2\|y\|^2$




In the parallelogram ABCD. the above equality means that $2AB^2 + 2AD^2 = AC^2 + BD^2$

9. Let $\beta$ be a basis for a finite dim inner product space.

a) If $\langle x, z \rangle = 0$, for all $z \in \beta$. then $x = 0$

b) If $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in \beta$, then $y = x$.

Solution: $\beta$ is a basis $\Rightarrow$ $x = \sum_{i=1}^{k} a_i z_i$, $z_i \in \beta$. $a_i$ scalars.

$\langle x, x \rangle = \langle x, \sum_{i=1}^{k} a_i z_i \rangle = \sum_{i=1}^{k} \bar{a}_i \langle x, z_i \rangle = 0$ $\Rightarrow$ $x = 0$.

$\langle x, z \rangle = \langle y, z \rangle$ $\Rightarrow$ $\langle x-y, z \rangle = 0$ for all $z \in \beta$

$\Rightarrow$ by a) $x - y = 0$ i.e. $x = y$.

13. Suppose that $\langle \cdot , \cdot \rangle_1$ and $\langle \cdot , \cdot \rangle_2$ are two inner products on a vector space $V$. Prove that $\langle \cdot , \cdot \rangle = \langle \cdot , \cdot \rangle_1 + \langle \cdot , \cdot \rangle_2$ is another inner product on $V$.

Solution:

- $\langle x+z, y \rangle = \langle x+z, y \rangle_1 + \langle x+z, y\rangle_2 = \langle x, y\rangle_1 + \langle z, y\rangle_1 + \langle x, y\rangle_2 + \langle z, y\rangle_2$
$= \langle x, y\rangle + \langle z, y\rangle$
- $\langle cx, y\rangle = \langle cx, y\rangle_1 + \langle cx, y\rangle_2 = c\langle x, y\rangle_1 + c\langle x, y\rangle_2 = c\langle x, y\rangle$
- $\overline{\langle x, y\rangle} = \overline{\langle x, y\rangle_1} + \overline{\langle x, y\rangle_2} = \langle y, x\rangle_1 + \langle y, x\rangle_2 = \langle y, x\rangle$
- $\langle x, x\rangle = \langle x, x\rangle_1 + \langle x, x\rangle_2 > 0$. if $x \neq 0$.

15. Prove that if $V$ is an inner product space, then $|\langle x, y\rangle| = \|x\| \cdot \|y\|$ iff one of the vectors $x$ or $y$ is a multiple of the other.

Solution: Assume $x \neq 0$, $y \neq 0$ and the identity holds.

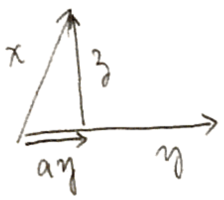
$a := \frac{\langle x, y\rangle}{\|y\|^2}$ $\qquad z := x - ay$.

$\langle y, z\rangle = 0$. $\qquad |a| = \frac{\|x\|}{\|y\|}$ (by assumption)

By parallelogram identity,

$2\|z\|^2 + 2\|ay\|^2 = 2\|x\|^2$

$\Rightarrow \|z\| = 0 \quad \Rightarrow \quad x = ay$




The inverse implication is easy.

16. a) Show that the vector space $H$ with $\langle \cdot , \cdot \rangle$ is an inner product space.

( $H$: the space of continuous complex-valued functions defined on $[0, 2\pi]$

$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\, dt$

b) $V = \mathbf{C}([0,1])$. $\langle f, g\rangle = \int_0^{1/2} f(t)\, g(t)\, dt$ Is this an inner product on $V$?

Solution: a)

- $\langle f+h, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} (f(t)+h(t))\,\overline{g(t)}\, dt = \langle f, g\rangle + \langle h, g\rangle$
- $\langle cf, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} c f(t)\, dt = c\langle f, g\rangle$
- $\overline{\langle f, g\rangle} = \overline{\frac{1}{2\pi}\int_0^{2\pi} f \bar{g}\, dt} = \langle g, f\rangle$
- $\langle f, f\rangle = \frac{1}{2\pi}\int_0^{2\pi} |f|^2\, dt > 0$ if $f$ is not zero.

b) $f(t) = \begin{cases} 0 & x \neq \frac{1}{2} \\ x - \frac{1}{2} & x > \frac{1}{2} \end{cases}$ $\langle f, f \rangle = 0$ but $f \neq 0$

17. $T$ a lin. operator on an inner product space $V$, and suppose that $\| T(x) \| = \| x \|$ for all $x$. Prove that $T$ is one-to-one.

Solution: If $T(x) = 0$. Then $\|x\| = \|T(x)\| = \|0\| = 0 \Rightarrow x = 0$ i.e.

$T$ is one-to-one

18. Let $V$ be a v.s. $/F$. $F = \mathbb{R}$ or $\mathbb{C}$. $W$ an inner product space over $F$ with inner product $\langle \cdot, \cdot \rangle$. If $T: V \to W$ is linear, prove that $\langle x, y \rangle' = \langle T(x), T(y) \rangle$ defines an inner product on $V$ iff $T$ is one-to-one.

Solution: $\Rightarrow$ If $\langle \cdot, \cdot \rangle'$ is an inner product, then $T(x) = 0 \Rightarrow$

$\langle x, x \rangle' = \langle T(x), T(x) \rangle = 0 \Rightarrow x = 0. \Rightarrow T$ is one-to-one.

$\Leftarrow$ • $\langle x + z, y \rangle' = \langle T(x+z), T(y) \rangle = \cdots = \langle x, y \rangle' + \langle z, y \rangle'$

• $\langle cx, y \rangle' = \langle T(cx), y \rangle = \cdots = \langle x, y \rangle'$

• $\overline{\langle x, y \rangle'} = \overline{\langle T(x), T(y) \rangle} = \langle y, x \rangle'$

• $\langle x, x \rangle' = \langle T(x), T(x) \rangle > 0$ if $x \neq 0$

(Note that for the last one, we use the condition that $T$ is 1-1.)