Topic: ◦ Revision Exercise

1) Recall: Laplacian of a function $f: \mathbb{R}^3 \to \mathbb{R}$

$\triangleq \Delta f = \nabla \cdot \nabla f = f_{xx} + f_{yy} + f_{zz}$.

A function f is harmonic if $\Delta f \equiv 0$.

We have proven two results last time:

a) Green's first identity:

$$\iint_S f \nabla g \cdot n \, dS = \iiint_D (f \Delta g + \nabla f \cdot \nabla g) \, dV.$$

b) Suppose $\Delta f \equiv 0$ on D and $f \equiv 0$ on S.
Then $f \equiv 0$ on D.

c) Deduce that if two functions f and g are harmonic on D and $f \equiv g$ on S, then $f \equiv g$ on D.

Ans:

Let $h = f - g$.

Then $\Delta h = \Delta f - \Delta g = 0$ and $h = 0$ on S.

By b), we have $h \equiv 0$. So $f \equiv g$.

d) Let f be a harmonic function.

Prove that

$$\frac{1}{a^2} \iint_{S_a} f \, dS = \frac{1}{b^2} \iint_{S_b} f \, dS,$$

in which S_r is the sphere of radius r around the origin.

(You may assume the fact that

$$\frac{d}{dr} \iint_{S_1} f(r) \, dS = \iint_{S_1} \frac{d}{dr} f(r) \, dS \text{)}$$

Ans: Let $F(r) = \frac{1}{r^2} \iint_{S_r} f(x,y,z)\,dS$.

Using spherical coordinates,

$$F(r) = \frac{1}{r^2} \int_0^{2\pi}\int_0^{\pi} f(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)\, r^2 \sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} f(r(\sin\phi\cos\theta), r(\sin\phi\sin\theta), r(\cos\phi))\sin\phi\, d\phi\, d\theta$$

$$= \iint_{S_1} f(rx, ry, rz)\, dS$$

$$\Rightarrow \frac{dF(r)}{dr} = \frac{d}{dr}\iint_{S_1} f(rx,ry,rz)\,dS$$

$$= \iint_{S_1} \frac{d}{dr} f(rx,ry,rz)\,dS$$

$$= \iint_{S_1} (x f_x + y f_y + z f_z)\,dS$$

$$= \iint_{S_1} \nabla f(rx,ry,rz)\cdot(x,y,z)\,dS$$

$$= \iint_{S_r} \nabla f(x,y,z)\cdot \vec{n}\,dS$$

$$\overset{\text{div}}{\underset{\text{thm}}{=}} \iiint_{B_r} (\nabla\cdot\nabla f)\,dV$$

$$= 0 \quad \text{as} \quad \Delta f = 0.$$

$\Rightarrow F(r) = \text{constant}$ and $F(a) = F(b)$ //

e) Prove the mean value property for harmonic functions:

$$f(p) = \frac{1}{4\pi a^2}\iint_{S_a(p)} f\,dS,$$

where $S_a(P)$ is the sphere of radius a around P.

Ans: Assume $P = 0$.

By d), $\frac{1}{b^2}\iint_{S_b} f\,dS = \frac{1}{a^2}\iint_{S_a} f\,dS$.

When b is very small, by continuity of f,
$f \approx f(0)$.

So $\frac{1}{b^2}\iint_{S_b} f\,dS \approx \frac{f(0)}{b^2}\iint_{S_b} dS = 4\pi f(0)$.

$\Rightarrow \quad f(0) = \frac{1}{4\pi a^2}\iint_{S_a} f\,dS$.

For $P \neq 0$, define $f_P(x,y,z) = f((x,y,z)+P)$.
Note that $\Delta f_P = \Delta f = 0$.

So $f_P(0) = \frac{1}{4\pi a^2}\iint_{S_a(0)} f_P\,dS$

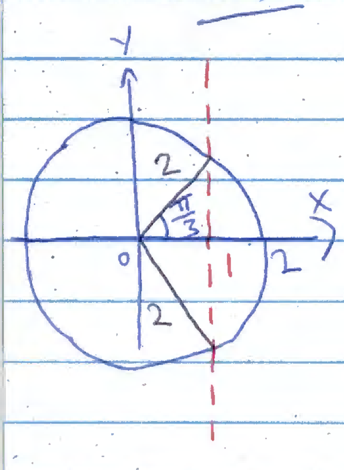
$\Rightarrow \quad f(P) = \frac{1}{4\pi a^2}\iint_{S_a(P)} f\,dS$,,

2) Let S be the part of the spherical surface $x^2+y^2+z^2=4$, lying in $x^2+y^2>1$.

a) Compute the outward flux through S of the vector field $\vec{F} = (-y, x, z)$.

b) Compute the volume between the cylinder $x^2+y^2=1$ & S.

Ans: a) The outward flux



$= \iint_S \vec{F}\cdot\vec{n}\,dS$

$= \iint_S (-y, x, z)\cdot\frac{(x,y,z)}{2}\,dS$

$= \frac{1}{2}\iint_S z^2\,dS$

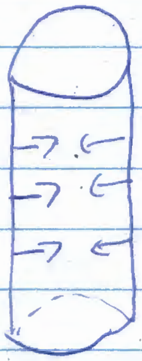
$= \frac{1}{2}\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (2\cos\phi)^2\, 4\sin\phi\,d\phi\,d\theta$

$$= 16\pi\left(-\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \cos^2\phi \; d\cos\phi\right)$$

$$= 4\sqrt{3}\,\pi_{\,//}$$

b) By div. thm,

$$\iiint_R \nabla\cdot\vec{F}\,dV = \iint_S \vec{F}\cdot\vec{n}\,dS + \iint_{\square} \vec{F}\cdot\vec{n}\,dS.$$



Note that $\iiint_R \nabla\cdot\vec{F}\,dV = \iiint_R dV = \mathrm{Vol}(R).$

$$\iint_S \vec{F}\cdot\vec{n}\,dS = 4\sqrt{3}\,\pi \text{ (by a)}.$$

$$\iint_{\square} \vec{F}\cdot\vec{n}\,dS = \iint_{\square} (-y, x, z)\cdot(-x, -y, 0)\,dA$$

$$= 0$$

So $\mathrm{Vol}(R) = 4\sqrt{3}\,\pi_{\,//}$.

3) Find the line integral of $\vec{F} = (-y^3 + \cos x,\ x^3,\ -z^3)$ along the curve $\vec{r}(t) = (\cos t,\ \sin t,\ 1-\cos t - \sin t),\ t\in[0, 2\pi]$.

Ans:

Note that $\nabla\times F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ -y^3+\cos x & x^3 & -z^3 \end{vmatrix}$

$$= (0,\ 0,\ 3x^2 - 3y^2)$$

$$\vec{F}(x,y,z) = (-y^3 + \cos x,\ x^3,\ -z^3)$$

$$= (-y^3,\ x^3,\ 0) + (\cos x,\ 0,\ -z^3)$$

$$\triangleq \vec{F}_1(x,y,z) + \vec{F}_2(x,y,z)$$

Note that $\nabla \times \vec{F_2} = 0$. Hence $\oint_C \vec{F} \, d\vec{r} = 0$.

For $\vec{F_1}$,

$$\oint_C \vec{F_1} \cdot d\vec{r} = \int_0^{2\pi} (-\sin^3 t, \cos^3 t, 0) \cdot (-\sin t, \cos t, 1+\sin t - \cos t) \, dt$$

$$= \int_0^{2\pi} (\sin^4 t + \cos^4 t) \, dt$$

$$= 2 \int_0^{2\pi} \cos^4 t \, dt$$

$$= 2 \int_0^{2\pi} \left(\frac{\cos 2t + 1}{2} \right)^2 dt$$

$$= 2 \int_0^{2\pi} \frac{\cos^2 2t + 2\cos 2t + 1}{4} \, dt$$

$$= \frac{1}{2} \left(\int_0^{2\pi} \cos^2 2t \, dt + 2\pi \right)$$

$$= \frac{1}{2} \int_0^{2\pi} \frac{\cos 4t + 1}{2} \, dt + \pi$$

$$= \frac{\pi}{2} + \pi$$

$$= \frac{3\pi}{2}$$

Hence $\oint_C \vec{F} \cdot d\vec{r} = \frac{3\pi}{2}$ //